Some Arithmetic, Algebraic and Combinatorial Aspects of Plane Binary Trees

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Part I: Arithmetic and Algebra over Plane Binary Trees

Extending Arithmetic:

reals $\to$ intervals $\to$ mapped partitions of interval

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4. – **by** exploiting the *algebraic structure of partitions formed by rooted-plane-binary (rpb) trees*
5. – **thereby** provide algorithms for several algebras and their inclusions over rpb tree partitions

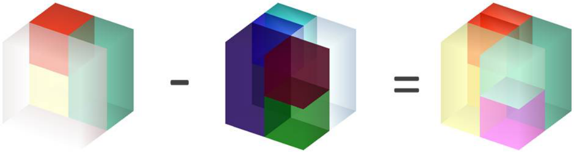

Figure: Arithmetic with coloured spaces.

arithmetic from intervals to their rpb-tree partitions

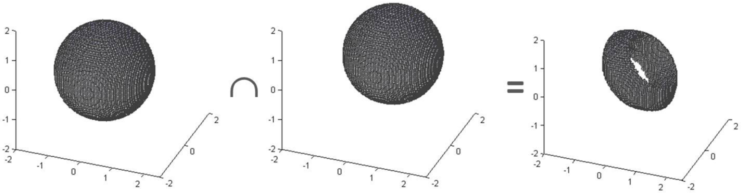

Figure: Intersection of two hollow spheres.

arithmetic from intervals to their rpb-tree partitions

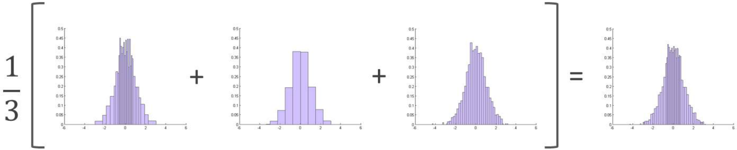

Figure: Histogram averaging.

An RP tree a root interval $\boldsymbol{x}_\rho \in \mathbb{IR}^d$

The regularly paved boxes of $\boldsymbol{x}_\rho$ can be represented by nodes of
rooted-plane-binary (rpb) trees of enumerative combinatorics
finite-rooted-binary (frb) trees of geometric group theory

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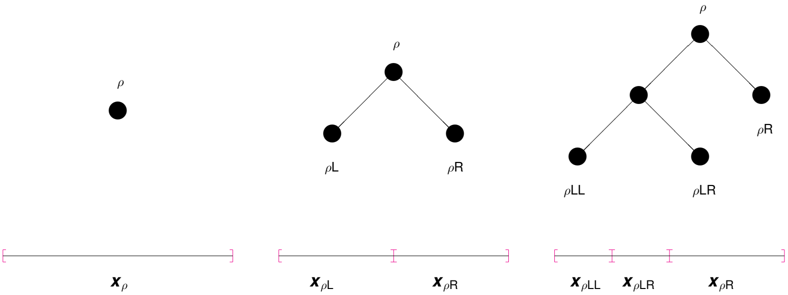
Leaf boxes of RP tree partition the root interval $\boldsymbol{x}_\rho \in \mathbb{IR}^1$



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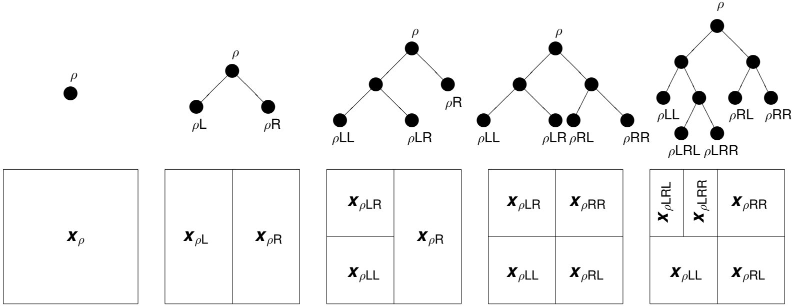

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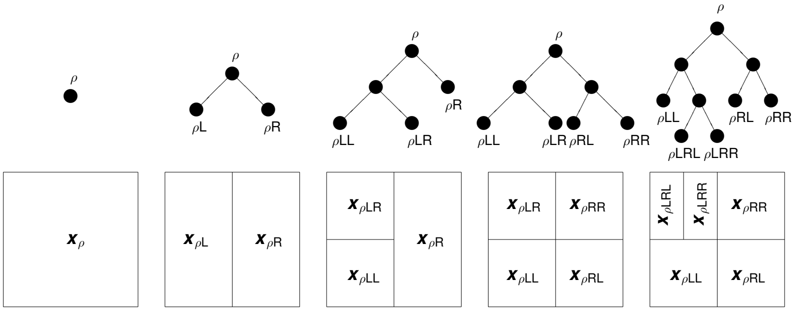
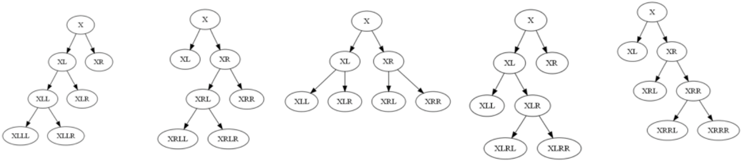


By this “RP Peano’s curve” rpb-trees encode paritions of $\boldsymbol{x}_\rho \in \mathbb{IR}^d$

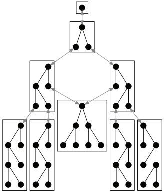
Algebraic Structure and Combinatorics of RPs

Leaf-depth encoded RPs

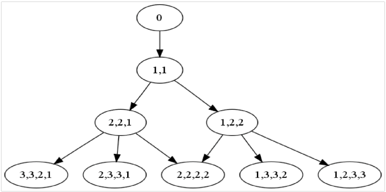


(3, 3, 2, 1) (1, 3, 3, 2) (2, 2, 2, 2) (2, 3, 3, 1) (1, 2, 3, 3)

There are C_k RPs with k splits

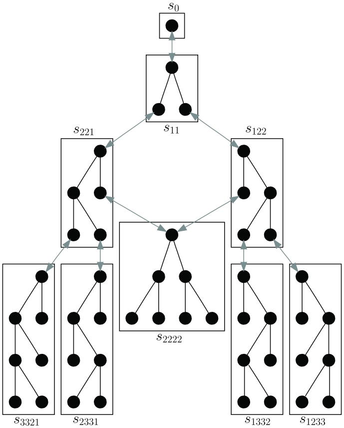


$$
\begin{aligned}
C_0 &= 1\\
C_1 &= 1\\
C_2 &= 2\\
C_3 &= 5\\
C_4 &= 14\\
C_5 &= 42\\
\dots &= \dots\\
C_k &= \frac{(2k)!}{(k+1)!k!}\\
\dots &= \dots\\
C_{15} &= 9694845\\
\dots &= \dots\\
C_{20} &= 6564120420\\
\dots &= \dots
\end{aligned}
$$



Hasse (transition) Diagram of Regular Pavings

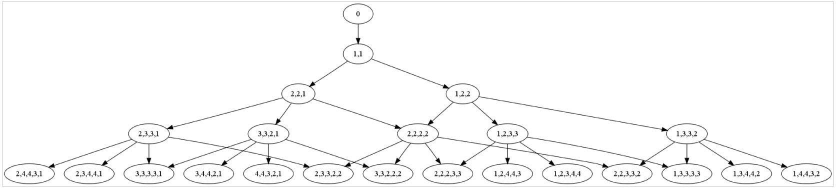
Transition diagram over $\mathbb{S}_{0:3}$ with split/reunion operations



RS, W.Taylor and G.Teng, Catalan Coefficients, Sequence A185155 in The On-Line Encyclopedia of Integer Sequences, 2012, http://oeis.org

Hasse (transition) Diagram of Regular Pavings

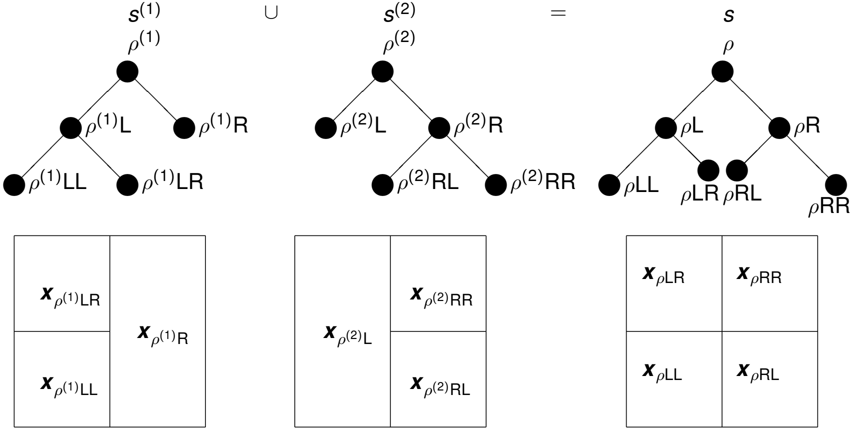
Transition diagram over $\mathbb{S}_{0:4}$ with split/reunion operations



1. The above state space is denoted by $\mathbb{S}_{0:4}$
2. Number of RPs with k splits is the Catalan number C_k
3. There is more than one way to reach a RP by k splits
4. Randomized enclosure algorithms are Markov chains on $\mathbb{S}_{0:\infty}$

RPs are closed under union operations

$s^{(1)} \cup s^{(2)} = s$ is union of two RPs $s^{(1)}$ and $s^{(2)}$ of $\boldsymbol{x}_\rho \in \mathbb{R}^2$.



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Lemma 1: The algebraic structure of rpb-trees (underlying Thompson's group) is closed under union operations.

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Proof: by a "transparency overlay process" argument (cf. Meier 2008).

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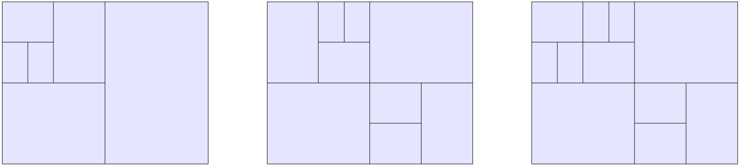

Algorithm 1: `RPUnion`$(\rho^{(1)}, \rho^{(2)})$

input : Root nodes $\rho^{(1)}$ and $\rho^{(2)}$ of RPs $s^{(1)}$ and $s^{(2)}$, respectively, with root box $\boldsymbol{x}_{\rho^{(1)}} = \boldsymbol{x}_{\rho^{(2)}}$

output : Root node ρ of RP $s = s^{(1)} \cup s^{(2)}$

if `IsLeaf`$(\rho^{(1)})$ & `IsLeaf`$(\rho^{(2)})$ **then**

  $\rho \leftarrow$ `Copy`$(\rho^{(1)})$

  return ρ

end

else if !`IsLeaf`$(\rho^{(1)})$ & `IsLeaf`$(\rho^{(2)})$ **then**

  $\rho \leftarrow$ `Copy`$(\rho^{(1)})$

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else if `IsLeaf`$(\rho^{(1)})$ & !`IsLeaf`$(\rho^{(2)})$ **then**

  $\rho \leftarrow$ `Copy`$(\rho^{(2)})$

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end

else

  !`IsLeaf`$(\rho^{(1)})$ & !`IsLeaf`$(\rho^{(2)})$

end

Make ρ as a node with $\boldsymbol{x}_\rho \leftarrow \boldsymbol{x}_{\rho^{(1)}}$

Graft onto ρ as left child the node `RPUnion`$(\rho^{(1)}\mathrm{L}, \rho^{(2)}\mathrm{L})$

Graft onto ρ as right child the node `RPUnion`$(\rho^{(1)}\mathrm{R}, \rho^{(2)}\mathrm{R})$

return ρ

Note: this is not the minimal union of the (Boolean mapped) RPs of Jaulin et. al. 2001

Dfn: Mapped Regular Paving (MRP)

- Let $s \in \mathbb{S}_{0:\infty}$ be an RP with root node ρ and root box $\boldsymbol{x}_\rho \in \mathbb{IR}^d$

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- Let $\mathbb{V}(s)$ and $\mathbb{L}(s)$ denote the sets all nodes and leaf nodes of s, respectively.
- Let $f : \mathbb{V}(s) \to \mathbb{Y}$ map each node of s to an element in $\mathbb{Y}$ as follows:

$$\{\rho \mathsf{v} \mapsto f_{\rho \mathsf{v}} : \rho \mathsf{v} \in \mathbb{V}(s), f_{\rho \mathsf{v}} \in \mathbb{Y}\} \ .$$

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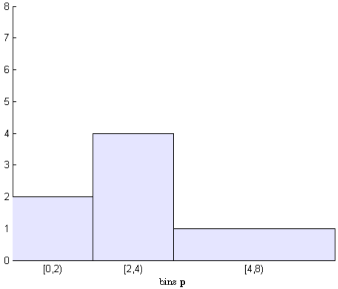
- Such a map f is called a *$\mathbb{Y}$-mapped regular paving ($\mathbb{Y}$-MRP)*.
- Thus, a $\mathbb{Y}$-MRP f is obtained by augmenting each node $\rho\mathsf{v}$ of the RP tree s with an additional data member $f_{\rho\mathsf{v}}$.

Examples of $\mathbb{Y}$-MRPs

If $\mathbb{Y} = \mathbb{R}$

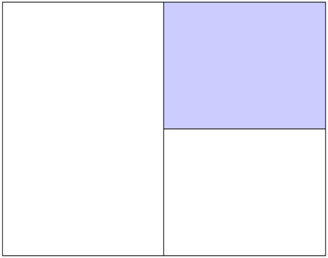
$\mathbb{R}$-MRP over s_{221} with $x_\rho = [0, 8]$



Examples of $\mathbb{Y}$-MRPs

If $\mathbb{Y} = \mathbb{B}$

$\mathbb{B}$-MRP over s_{122} with $x_\rho = [0, 1]^2$ (e.g. Jaulin et. al. 2001)

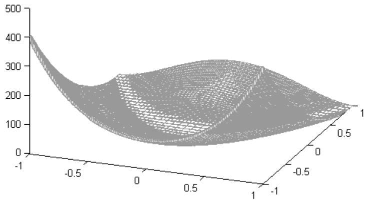


Examples of $\mathbb{Y}$-MRPs

If $\mathbb{Y} = \mathbb{IR}$

– rpb tree representation for interval inclusion algebra

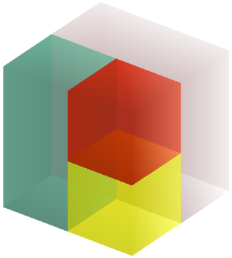
$\mathbb{IR}$-MRP enclosure of the Rosenbrock function with $x_\rho = [-1, 1]^2$



Examples of $\mathbb{Y}$-MRPs

If $\mathbb{Y} = [0, 1]^3$
– R G B colour maps

$[0, 1]^3$-MRP over s_{3321} with $x_\rho = [0, 1]^3$

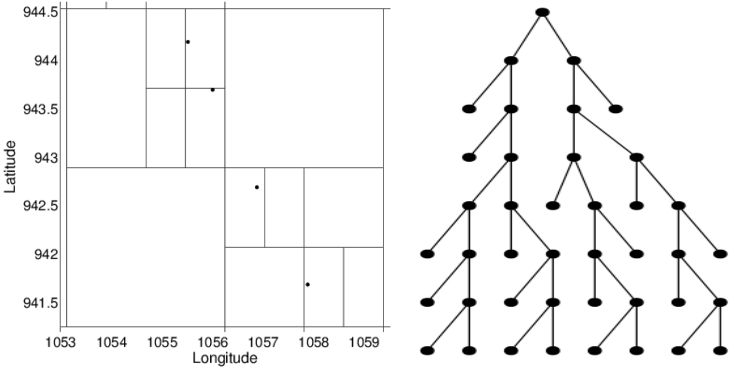


Examples of $\mathbb{Y}$-MRPs

If $\mathbb{Y} = \mathbb{Z}_+ := \{0, 1, 2, ...\}$

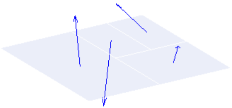
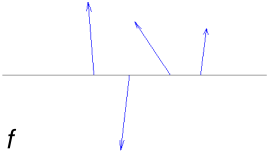
– radar-measured aircraft trajectory data

$\mathbb{Z}_+$-MRP trajectory of an aircraft and its tree

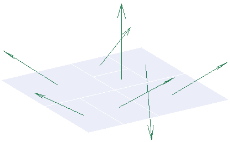
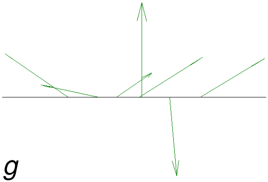


Examples of $\mathbb{Y}$-MRPs

If $\mathbb{Y} = \mathbb{S}^2$, $\boldsymbol{x}_\rho = [0,1]^2$ – vector-MRPs

Two Views of f

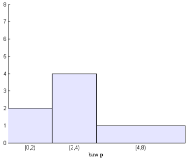



Two Views of g

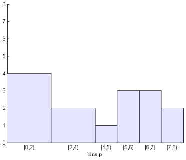
$\mathbb{Y}$-MRP Arithmetic

If $\star : \mathbb{Y} \times \mathbb{Y} \to \mathbb{Y}$ then we can extend $\star$ point-wise to two $\mathbb{Y}$-MRPs *f* and *g* with root nodes $\rho^{(1)}$ and $\rho^{(2)}$ via `MRPOperate`$(\rho^{(1)}, \rho^{(2)}, \star)$.
This is done using `MRPOperate`$(\rho^{(1)}, \rho^{(2)}, +)$

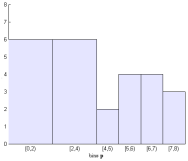
f



g

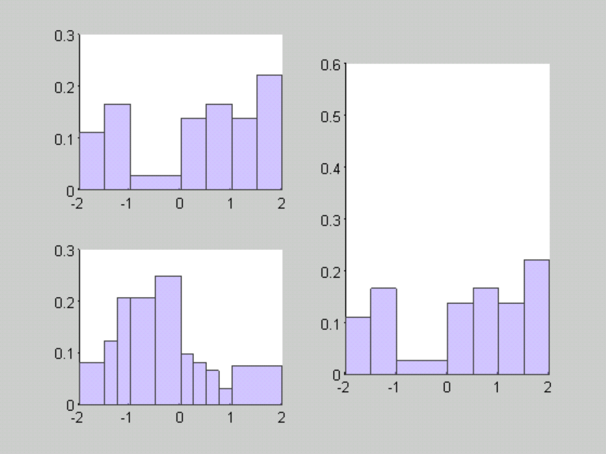


$f + g$



$\mathbb{R}$-MRP Addition by $\texttt{MRPOperate}(\rho^{(1)}, \rho^{(2)}, +)$

adding two piece-wise constant functions or $\mathbb{R}$-MRPs



Algorithm 2: $\texttt{MRPOperate}(\rho^{(1)}, \rho^{(2)}, \star)$

input : two root nodes $\rho^{(1)}$ and $\rho^{(2)}$ with same root box $\boldsymbol{x}_{\rho^{(1)}} = \boldsymbol{x}_{\rho^{(2)}}$ and binary operation $\star$.

output : the root node ρ of $\mathbb{Y}$-MRP $h = f \star g$.

Make a new node ρ with box and image
$\boldsymbol{x}_\rho \leftarrow \boldsymbol{x}_{\rho^{(1)}}; h_\rho \leftarrow f_{\rho^{(1)}} \star g_{\rho^{(2)}}$

if $\texttt{IsLeaf}(\rho^{(1)})$ & $!\texttt{IsLeaf}(\rho^{(2)})$ **then**
  Make temporary nodes L′, R′
  $\boldsymbol{x}_{\mathrm{L}'} \leftarrow \boldsymbol{x}_{\rho^{(1)}\mathrm{L}}; \boldsymbol{x}_{\mathrm{R}'} \leftarrow \boldsymbol{x}_{\rho^{(1)}\mathrm{R}}$
  $f_{\mathrm{L}'} \leftarrow f_{\rho^{(1)}}, f_{\mathrm{R}'} \leftarrow f_{\rho^{(1)}}$
  Graft onto ρ as left child the node $\texttt{MRPOperate}(\mathrm{L}', \rho^{(2)}\mathrm{L}, \star)$
  Graft onto ρ as right child the node $\texttt{MRPOperate}(\mathrm{R}', \rho^{(2)}\mathrm{R}, \star)$
end

else if $!\texttt{IsLeaf}(\rho^{(1)})$ & $\texttt{IsLeaf}(\rho^{(2)})$ **then**
  Make temporary nodes L′, R′
  $\boldsymbol{x}_{\mathrm{L}'} \leftarrow \boldsymbol{x}_{\rho^{(2)}\mathrm{L}}; \boldsymbol{x}_{\mathrm{R}'} \leftarrow \boldsymbol{x}_{\rho^{(2)}\mathrm{R}}$
  $g_{\mathrm{L}'} \leftarrow g_{\rho^{(2)}}, g_{\mathrm{R}'} \leftarrow g_{\rho^{(2)}}$
  Graft onto ρ as left child the node $\texttt{MRPOperate}(\rho^{(1)}\mathrm{L}, \mathrm{L}', \star)$
  Graft onto ρ as right child the node $\texttt{MRPOperate}(\rho^{(1)}\mathrm{R}, \mathrm{R}', \star)$
end

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end
return ρ

Unary transformations are easy too

Let `MRPTransform`(ρ, τ) apply the unary transformation $\tau : \mathbb{R} \to \mathbb{R}$ to a given $\mathbb{R}$-MRP f with root node ρ as follows:

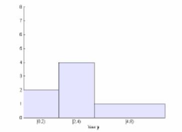
- copy f to g
- recursively set $f_{\rho v} = \tau(f_{\rho v})$ for each node ρv in g
- return g as $\tau(f)$

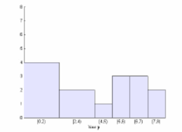
Minimal Representation of $\mathbb{R}$-MRP

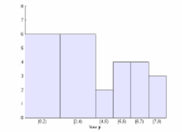
Algorithm 3: `MinimiseLeaves`(ρ)

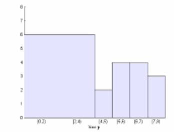
input : ρ, the root node of $\mathbb{R}$-MRP f.
output : Modify f into $\leftthreetimes(f)$, the unique $\mathbb{R}$-MRP with fewest leaves.

if !`IsLeaf`(ρ) **then**
  `MinimiseLeaves`$(\rho \mathsf{L})$
  `MinimiseLeaves`$(\rho \mathsf{R})$
  if `IsCherry`(ρ) & $(f_{\rho \mathsf{L}} = f_{\rho \mathsf{R}})$ **then**
    $f_\rho \leftarrow f_{\rho \mathsf{L}}$
    `Prune`$(\rho \mathsf{L})$
    `Prune`$(\rho \mathsf{R})$
  end
end


(a) f


(b) g


(c) $f+g$


(d) $\leftthreetimes(f+g)$

Thus, we can obtain arithmetical expressions specified by finitely many sub-expressions in a **directed acyclic graph** whose:

Arithmetic and Algebra of $\mathbb{R}$-MRPs

Thus, we can obtain arithmetical expressions specified by finitely many sub-expressions in a **directed acyclic graph** whose:

- inputs and output **nodes** are themselves $\mathbb{R}$-MRPs

Arithmetic and Algebra of $\mathbb{R}$-MRPs

Thus, we can obtain arithmetical expressions specified by finitely many sub-expressions in a **directed acyclic graph** whose:

- inputs and output **nodes** are themselves $\mathbb{R}$-MRPs
- and whose **edges** involve:

Arithmetic and Algebra of $\mathbb{R}$-MRPs

Thus, we can obtain arithmetical expressions specified by finitely many sub-expressions in a **directed acyclic graph** whose:

- inputs and output **nodes** are themselves $\mathbb{R}$-MRPs
- and whose **edges** involve:
  1. a binary arithmetic operation $\star \in \{+, -, \cdot, /\}$ over two $\mathbb{R}$-MRPs,
  2. a standard transformation of $\mathbb{R}$-MRP by elements of $\mathfrak{S} := \{\exp, \sin, \cos, \tan, \ldots\}$ and
  3. their compositions.

Stone-Weierstrass Theorem: $\mathbb{R}$-MRPs Dense in $C(\boldsymbol{x}_\rho, \mathbb{R})$

Theorem

Let $\mathcal{F}$ be the class of $\mathbb{R}$-MRPs with the same root box $\boldsymbol{x}_\rho$. Then $\mathcal{F}$ is dense in $C(\boldsymbol{x}_\rho, \mathbb{R})$, the algebra of real-valued continuous functions on $\boldsymbol{x}_\rho$.

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Proof:

Since $\boldsymbol{x}_\rho \in \mathbb{IR}^d$ is a compact Hausdorff space, by the Stone-Weierstrass theorem we can establish that $\mathcal{F}$ is dense in $C(\boldsymbol{x}_\rho, \mathbb{R})$ with the topology of uniform convergence, provided that $\mathcal{F}$ is a sub-algebra of $C(\boldsymbol{x}_\rho, \mathbb{R})$ that separates points in $\boldsymbol{x}_\rho$ and which contains a non-zero constant function.

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We will show all these conditions are satisfied by $\mathcal{F}$

Stone-Weierstrass Theorem Contd.: $\mathbb{R}$-MRPs Dense in $C(\boldsymbol{x}_\rho, \mathbb{R})$

- $\mathcal{F}$ is a sub-algebra of $C(\boldsymbol{x}_\rho, \mathbb{R})$ since it is closed under addition and scalar multiplication.

Stone-Weierstrass Theorem Contd.: $\mathbb{R}$-MRPs Dense in $C(\boldsymbol{x}_\rho, \mathbb{R})$

- $\mathcal{F}$ is a sub-algebra of $C(\boldsymbol{x}_\rho, \mathbb{R})$ since it is closed under addition and scalar multiplication.
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- $\mathcal{F}$ is a sub-algebra of $C(\boldsymbol{x}_\rho, \mathbb{R})$ since it is closed under addition and scalar multiplication.
- $\mathcal{F}$ contains non-zero constant functions
- Finally, RPs can clearly separate distinct points $x, x' \in \boldsymbol{x}_\rho$ into distinct leaf boxes by splitting deeply enough.

Stone-Weierstrass Theorem Contd.: $\mathbb{R}$-MRPs Dense in $C(\boldsymbol{x}_\rho, \mathbb{R})$

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Stone-Weierstrass Theorem Contd.: $\mathbb{R}$-MRPs Dense in $C(\boldsymbol{x}_\rho, \mathbb{R})$

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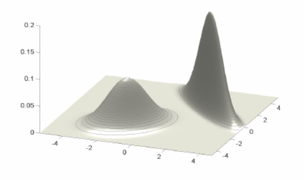
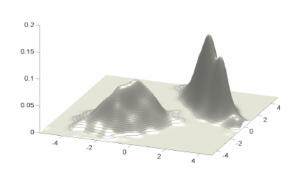
- Q.E.D.

Approximating Kernel Density Estimates by $\mathbb{R}$-MRPs

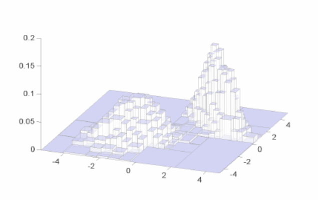


(a) True density.

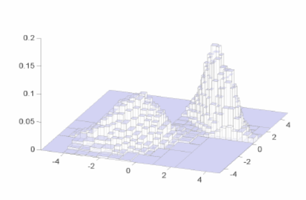


(c) MCMC bandwidth KDE.

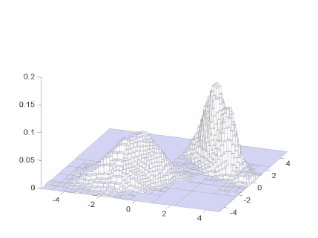
Approximating Kernel Density Estimates by $\mathbb{R}$-MRPs



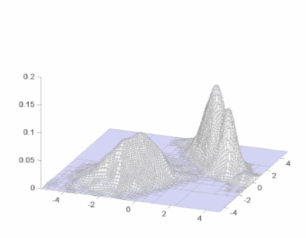
(a) $\overline{\psi} = 0.001$ (187 leaves).



(b) $\overline{\psi} = 0.005$ (316 leaves).



(c) $\overline{\psi} = 0.0001$ (919 leaves).



(d) $\overline{\psi} = 0.00001$ (4420 leaves).

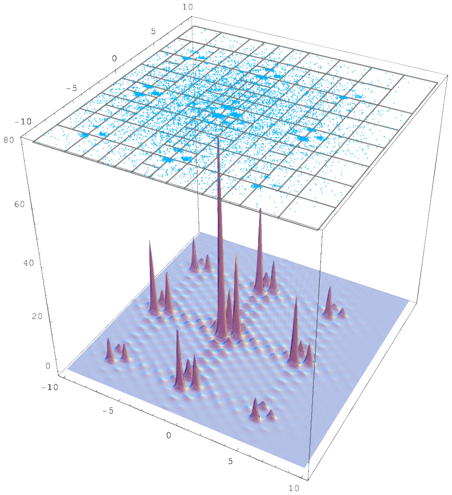
Approximating Kernel Density Estimates by $\mathbb{R}$-MRPs

Table J.4: 5-d case: estimated errors for KDE and RMRP-KDE approximations.

	$\hat{d}_{KL}$	$\hat{L}_1$ error	Time (s)	Leaves
KDE ($n_K = 2,000$)	0.41	0.66	7,350–8,880	n/a
RMRP-KDE approximations				
$\overline{\psi} = 0.0001$	5.06	0.96	1.0	$2,363$
$\overline{\psi} = 0.00005$	4.85	0.91	2.3	$4,639$
$\overline{\psi} = 0.00001$	4.51	0.85	8.7	$17,759$
$\overline{\psi} = 0.000005$	4.49	0.84	17.2	$31,335$
$\overline{\psi} = 0.000001$	3.33	0.76	66.1	$133,493$
$\overline{\psi} = 0.0000005$	3.31	0.75	131.0	$237,561$
$\overline{\psi} = 0.0000001$	3.54	0.74	470.0	$895,012$

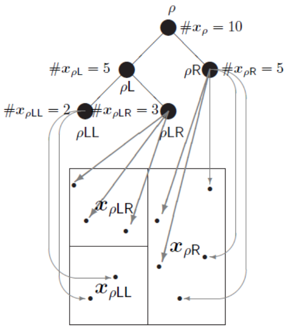
Nonparametric Density Estimation

Problem: Take samples from an unknown density f and consistently reconstruct f

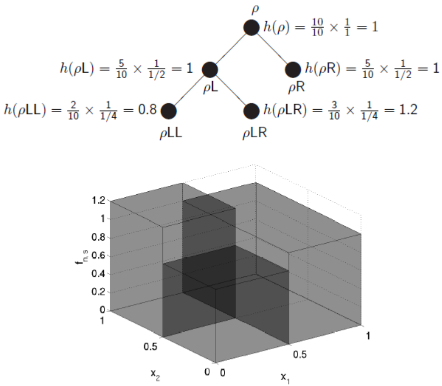


Nonparametric Density Estimation

Approach: Use statistical regular paving to get $\mathbb{R}$-MRP data-adaptive histogram



(a) An SRP tree and its constituents.

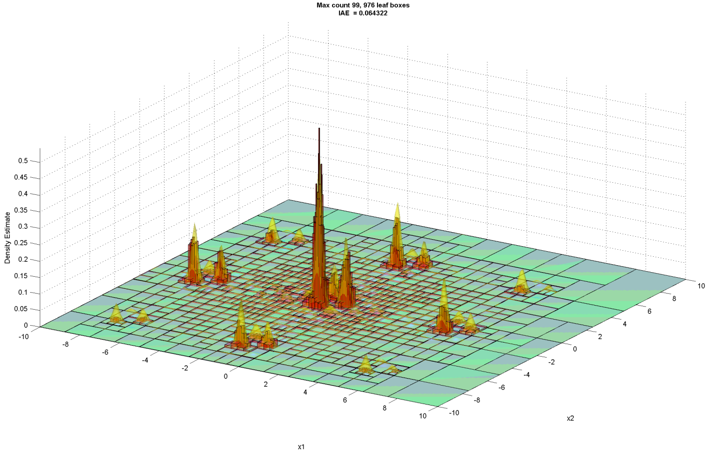


(b) An SRP histogram and its tree.

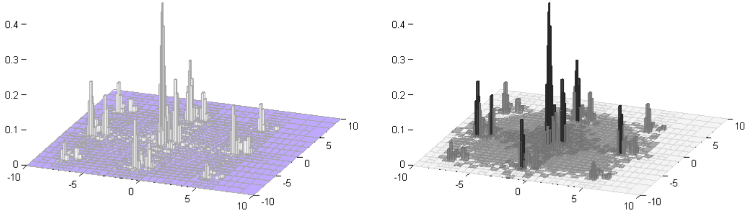
Nonparametric Density Estimation

Solution: $\mathbb{R}$-MRP histogram averaging allows us to produce a consistent Bayesian estimate of the density (up to 10 dimensions)

(Teng, Harlow, Lee and S., *ACM Trans. Mod. & Comp. Sim.*, [r. 2] 2012)

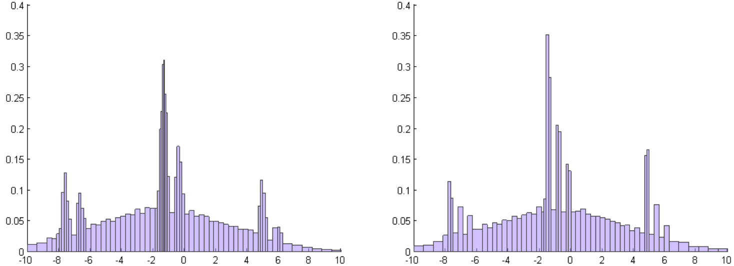


Coverage, Marginal & Slice Operators of $\mathbb{R}$-MRP



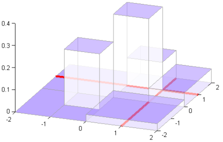
$\mathbb{R}$-MRP approximation to Levy density and its coverage regions with $\alpha = 0.9$ (light gray), $\alpha = 0.5$ (dark gray) and $\alpha = 0.1$ (black)

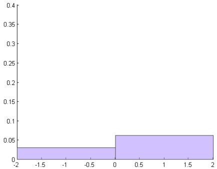
Coverage, Marginal & Slice Operators of $\mathbb{R}$-MRP

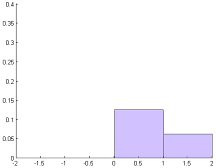


Marginal densities $f^{\{1\}}(x_1)$ and $f^{\{2\}}(x_2)$ along each coordinate of $\mathbb{R}$-MRP approximation

Coverage, Marginal & Slice Operators of ℝ-MRP



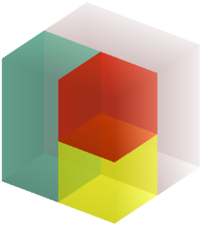




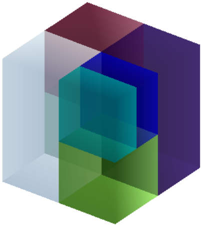
The slices of a simple ℝ-MRP in 2D

— “non-parametric regression arithmetic”

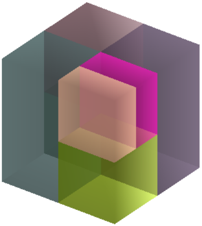
$[0,1]^3$-MRP Arithmetic over Colored Cubes



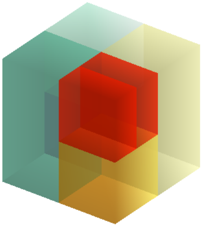
f



g

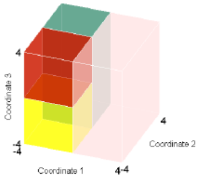


$f + g$

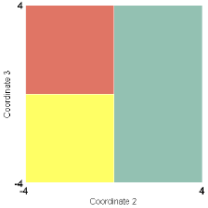


$f - g$

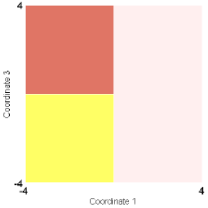
Slices of colored cube



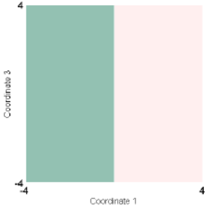
(a) $[0,1]^3$-MRP f



(b) $f^{|(x'_1)=(-2.0)}$



(c) $f^{|(x'_2)=(-2.0)}$



(d) $f^{|(x'_2)=(2.0)}$

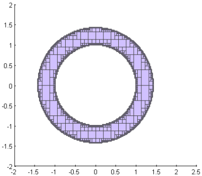
(e) $f^{|(x'_1,x'_3)=(-2.0,-2.0)}$

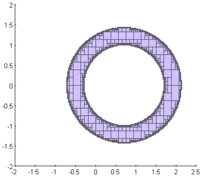
(f) $f^{|(x'_1,x'_3)=(-2.0,2.0)}$

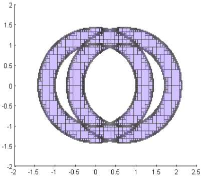
$\mathbb{B}$-MRP arithmetic – contractors, propagators & collaborators (Comp-aided Proofs in Anal./Dyn.)

Two Boolean-mapped regular pavings A_1 and A_2 and Boolean arithmetic operations with $+$ for set union, $-$ for symmetric set difference, $\times$ for set intersection, and $\div$ for set difference.

A_1 A_2 $A1 + A2$



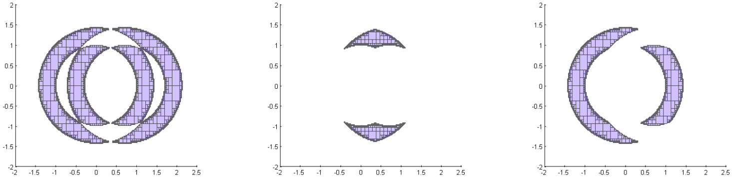




$\mathbb{B}$-MRP arithmetic – contractors, propagators & collaborators (Comp-aided Proofs in Anal./Dyn.)

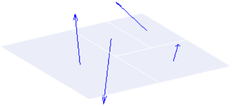
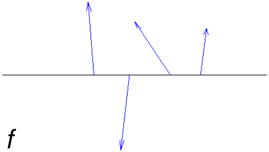
Two Boolean-mapped regular pavings A_1 and A_2 and Boolean arithmetic operations with $+$ for set union, $-$ for symmetric set difference, $\times$ for set intersection, and $\div$ for set difference.

$A_1 - A2$ $\qquad$ $A_1 \times A_2$ $\qquad$ $A_1 \div A_2$

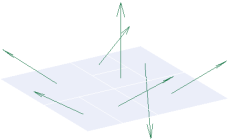
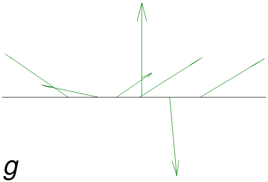


Vector-MRP arithmetic

If $\mathbb{Y} = \mathbb{S}^2$, $\boldsymbol{x}_\rho = [0,1]^2$ – vector-MRPs

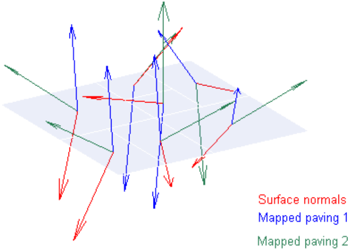
Two Views of *f*

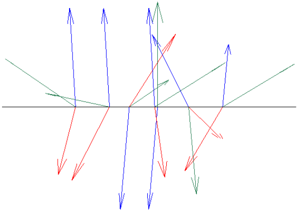



Two Views of *g*

Vector-MRP arithmetic

$f \times g$ — cross-product of vector-MRPs

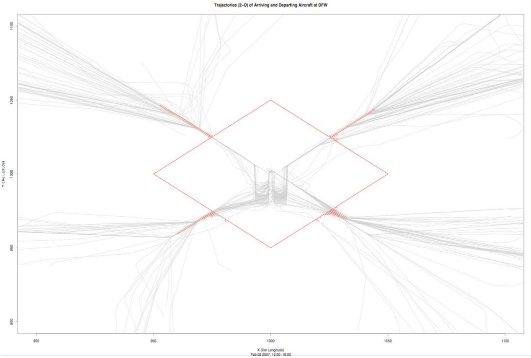




Air Traffic “Arithmetic”

(G. Teng, K. Kuhn and RS, *J. Aerospace Comput., Inf. & Com.*, 9:1, 14–25, 2012.)

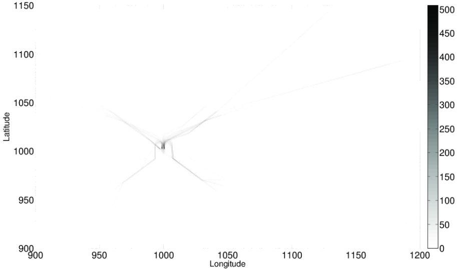
On a Good Day



Air Traffic “Arithmetic”

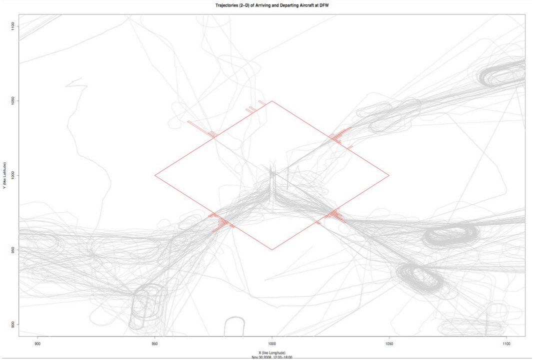
(G. Teng, K. Kuhn and RS, *J. Aerospace Comput., Inf. & Com.*, 9:1, 14–25, 2012.)

$\mathbb{Z}_+$-MRP On a Good Day



(G. Teng, K. Kuhn and RS, *J. Aerospace Comput., Inf. & Com.*, 9:1, 14–25, 2012.)

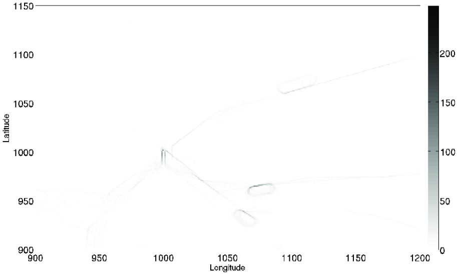
On a Bad Day



Air Traffic "Arithmetic"

(G. Teng, K. Kuhn and RS, *J. Aerospace Comput., Inf. & Com.*, 9:1, 14–25, 2012.)

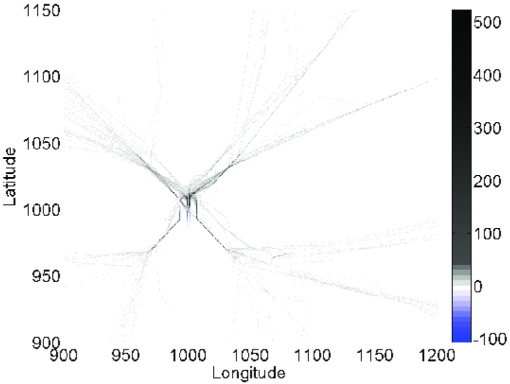
$\mathbb{Z}_+$-MRP On a Bad Day



Air Traffic “Arithmetic”

(G. Teng, K. Kuhn and RS, *J. Aerospace Comput., Inf. & Com.*, 9:1, 14–25, 2012.)

$\mathbb{Z}$-MRP pattern for Good Day $-$ Bad Day



MRS 1.0: A C++ Class Library for Statistical Set Processing, Harlow, S & York, 2013

MRS 1.0 is GNU auto-confiscated, Doxygenized, GPL-licensed (builds on GNU Sci. Lib., C-XSC & Boost) and has:

54 directories, 432 files

Language	files	blank	comment	code
C++	148	13274	10488	38738
C/C++ Header	123	8366	19634	9012
Bourne Shell	6	1137	1191	7872
MATLAB	14	522	103	1585
m4	3	149	20	1237
CSS	1	173	55	721
make	51	106	72	493
HTML	2	1	18	46
Bourne Again Shell	2	0	2	4
SUM:	350	23728	31583	59708

Its is templatised and can be extended to general $\mathbb{Y}$-MRPs.

Conclusions of Part I

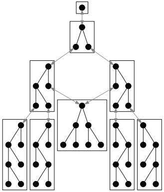
- $\mathbb{Y}$-MRPs provide rpb-tree partition arithmetic
- $\mathbb{IY}$-MRPs allow efficient arithmetic for Neumaier's inclusion algebras
-
  - $\mathbb{IY}$ can be $\mathbb{IR}$ for $\boldsymbol{f} : \mathbb{IR}^d \rightarrow \mathbb{IR}$
  - $\mathbb{IY}$ can be $\mathbb{IR}^m$ for $\boldsymbol{f} : \mathbb{IR}^d \rightarrow \mathbb{IR}^m$
  - $\mathbb{IY}$ can be $(\mathbb{IR}, \mathbb{IR}^m, \mathbb{IR}^{m^2})$ for range, gradient & Hessian of $\boldsymbol{f} : \mathbb{IR}^d \rightarrow \mathbb{IR}$
- Other obvious extensions include arithmetic over Taylor polynomial inclusion algebras
- In general the domain and range of $\boldsymbol{f}$ can be complete lattices with intervals and bisection operations
- We have seen several statistical applications of $\mathbb{Y}$-MRPs
- CODE: *mrs: a C++ class library for statistical set processing* by Harlow, S and York.

Part II: Combinatorics for Distributions over Plane Binary Trees

How many distinct "splitting paths" are there from the root node to a given rpb-tree T?
Let this be $B(T)$, the *Catalan Coefficient* of T.

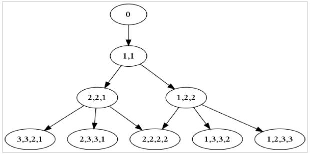
There are C_k RPs with k splits

and $k!$ distinct paths to them



1 1 2 1 1

k	C_k	$k!$
0	1	1
1	1	1
2	2	2
3	5	6
4	14	24
5	42	120
...	...	...
k	$\frac{(2k)!}{(k+1)!k!}$	$k!$
...	...	...



1 1 2 1 1

For example: $B((2,2,2,2)) = 2$ and $B((3,3,2,1)) = 1$

Catalan coefficients – OEIS A185155

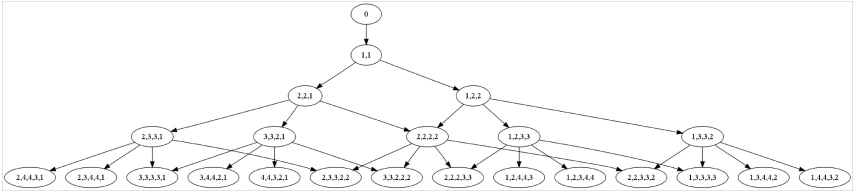
oeis.org/A185155

A185155 Catalan Coefficients.

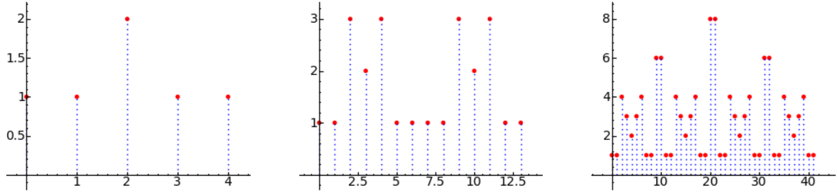
1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 3, 2, 3, 1, 1, 1, 1, 3, 2, 3, 1, 1, 1, 1, 4, 3, 2, 3, 4,
1, 1, 6, 6, 1, 1, 4, 3, 2, 3, 4, 1, 1, 8, 8, 1, 1, 4, 3, 2, 3, 4, 1, 1, 6, 6, 1, 1, 4, 3,
2, 3, 4, 1, 1 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,7

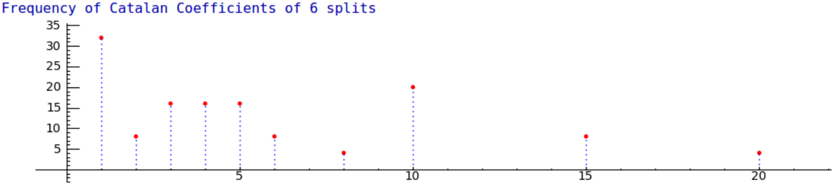
```
k     C_k      k!     (B_{k,0},B_{k,1},...,B_{k,C_k-1})
0     1        1      1
1     1        1      1
2     2        2      1, 1
3     5        6      1, 1, 2, 1, 1
4     14       24    1, 1, 3, 2, 3, 1, 1, 1, 1, 3, 2, 3, 1, 1
5     42       120  1, 1, 4, 3, 2, 3, 4, 1, 1, 6, 6, 1, 1, 4, 3, 2, 3, 4, 1,
   1, 8, 8, 1, 1, 4, 3, 2, 3, 4, 1, 1, 6, 6, 1, 1, 4, 3, 2, 3, 4, 1, 1
6     132     720   ...
.  .
```

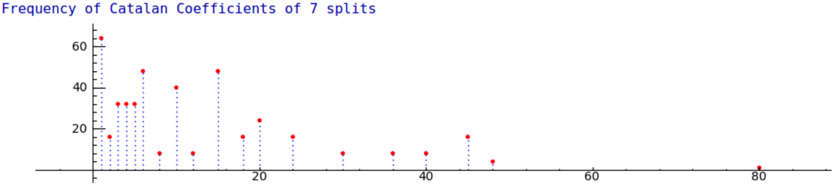


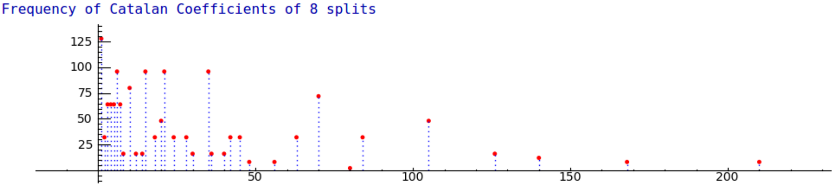
Catalan coefficients of rpb-trees with $3, 4, 5$ splits



Frequency of Catalan coefficients of rpb-trees with $6, 7, 8$ splits







Catalan coefficient of a rpb-tree

Let an interior node of T include the root node and exclude all leaf nodes of T. Then the Catalan coefficient of T is:

$$B(T) = \frac{|T|!}{\prod_{v \in T} |T_v|} = \frac{(\#\text{ of interior nodes of } T)!}{\prod_{v \in T} \#\text{ of interior nodes of sub-tree of } T \text{ with root } v}$$

Proof:

Catalan coefficient of a rpb-tree

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Proof:

Let $L(T_v)$ and $R(T_v)$ be left and right sub-trees of T_v with root node v. Then the number of distinct binary inter-leavings between the interior (split) nodes of $L(T_v)$ and $R(T_v)$ is:

$$\begin{aligned}
\binom{|L(T_v)| + |R(T_v)|}{|L(T_v)|} &= \frac{(|L(T_v)| + |R(T_v)|)!}{|L(T_v)|! \times |R(T_v)|!} \\
&= \frac{|T_v| \times (|L(T_v)| + |R(T_v)|)!}{|T_v| \times |L(T_v)|! \times |R(T_v)|!} \\
&= \frac{|T_v|!}{|T_v| \times |L(T_v)|! \times |R(T_v)|!}
\end{aligned}$$

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Proof:

And the number of distinct binary inter-leavings between the interior (split) nodes of $L(T_v)$ and $R(T_v)$ as well as their sub-trees and their sub-sub-trees and so on is:

$$\frac{|T_v|!}{|T_v| \times \cancel{|L(T_v)|!} \times \cancel{|R(T_v)|!}} \times \frac{\cancel{|L(T_v)|!}}{|L(T_v)| \times |L(L(T_v))|! \times |R(L(T_v))|!} \times \frac{\cancel{|R(T_v)|!}}{|R(T_v)| \times |L(R(T_v))|! \times |R(R(T_v))|!} \times \cdots \frac{1!}{1 \times 0! \times 0!}$$

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Catalan coefficient of a rpb-tree

Let an interior node of T include the root node and exclude all leaf nodes of T. Then the Catalan coefficient of T is:

$$B(T) = \frac{|T|!}{\prod_{v \in T} |T_v|} = \frac{(\#\text{ of interior nodes of } T)!}{\prod_{v \in T} \#\text{ of interior nodes of sub-tree of } T \text{ with root } v}$$

Proof:

$$\begin{aligned} B(T_v) &= \frac{|T_v|!}{|T_v| \times |L(T_v)|! \times |R(T_v)|!} \times B(L(T_v)) \times B(R(T_v)) \\ &= \frac{|T_v|!}{|T_v| \times |L(T_v)| \times |R(T_v)| \times |L(L(T_v))| \times |R(L(T_v))| \times \cdots} \\ &= \frac{|T_v|!}{\prod_{u \in T_v} |T_u|} \end{aligned}$$

Therefore,

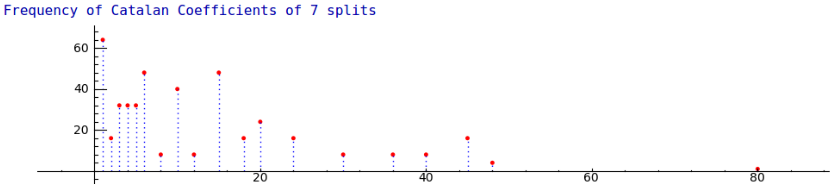
$$B(T) = \frac{|T|!}{\prod_{v \in T} |T_v|}$$

An Example Catalan coefficient calculation

Consider the perfectly balanced tree $(3,3,3,3,3,3,3,3)$ with $k = 7$ splits and 8 leaves (all with depth 3).

Then $B((3,3,3,3,3,3,3,3))$

$$= \frac{7!}{7 \times 3 \times 3 \times 1 \times 1 \times 1 \times 1} = \frac{\overset{2}{\cancel{6}} \times 5 \times 4 \times \cancel{3} \times 2}{\cancel{3} \times \cancel{3}} = 80.$$



Planar Yule Model

Model (Planar Yule)

A rpb-tree with k splits is obtained from one with $k-1$ splits by splitting one of the k leaves uniformly at random.

$$\Pr(T) = B(T)\frac{1}{k!}$$

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This model also gives:

- the probability of a binary search tree with random input given by the unifrom distribution on all permutations of $[k]$ (Fill, 1994)
- the probability of an planar Yule tree (non-planar case is the Yule speciation model in Phylogenetics)

f-Splitting Model

Model (f-Splitting)

Let f be a probability density function on $[0,1]$. We can represent the rpb-tree by the corresponding dyadic partition of $[0,1]$. We split a node v of an rpb-tree corresponding to the interval $[\underline{v},\overline{v}] = [2^{-i}, 2^{-j}]$ with probability $\int_{\underline{v}}^{\overline{v}} f(x)dx$.

$$\Pr(T) = B(T) \prod_{v \in T} \left(\int_{\underline{v}}^{\overline{v}} f(x)dx \right)$$

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$$\Pr(T) = B(T) \prod_{v \in T} \left(\int_{\underline{v}}^{\overline{v}} f(x)dx \right)$$

This model also gives:

- the Statistically equivalent block rule in density estimation
- gives a flexible way to explore the tree shapes, eg. $f \sim Beta(\alpha, \beta)$

Split-Path Invariant Distributions

Model (Split-Path Invariant)

$$\Pr(T) = B(T) \times \Pr\{\textit{any split-path to } T \textit{ from root node}\}$$

Split-Path Invariant Distributions

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The Planar Yule Model and f-Splitting Model are examples of Split-Path Invariant Distributions. Such distributions on rpb-trees can also be used to model:

- infection trees in epidemics on various hidden contact graphs (eq. all-to-all, one-super-hub, etc.)
- causal trees underlying self-exciting point processes
- by counting planar tree paths that map to non-palanar tree paths we can induce these distributions on non-planar trees

Conclusions of Part II

- Several probablistic models on rooted planar binary trees exist
- Using the notion of split-path invariance and Catalan coefficients we can specify various useful distributions on rpb-trees
- These distributions can also be further projected onto non-planar binary trees

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Thank you!