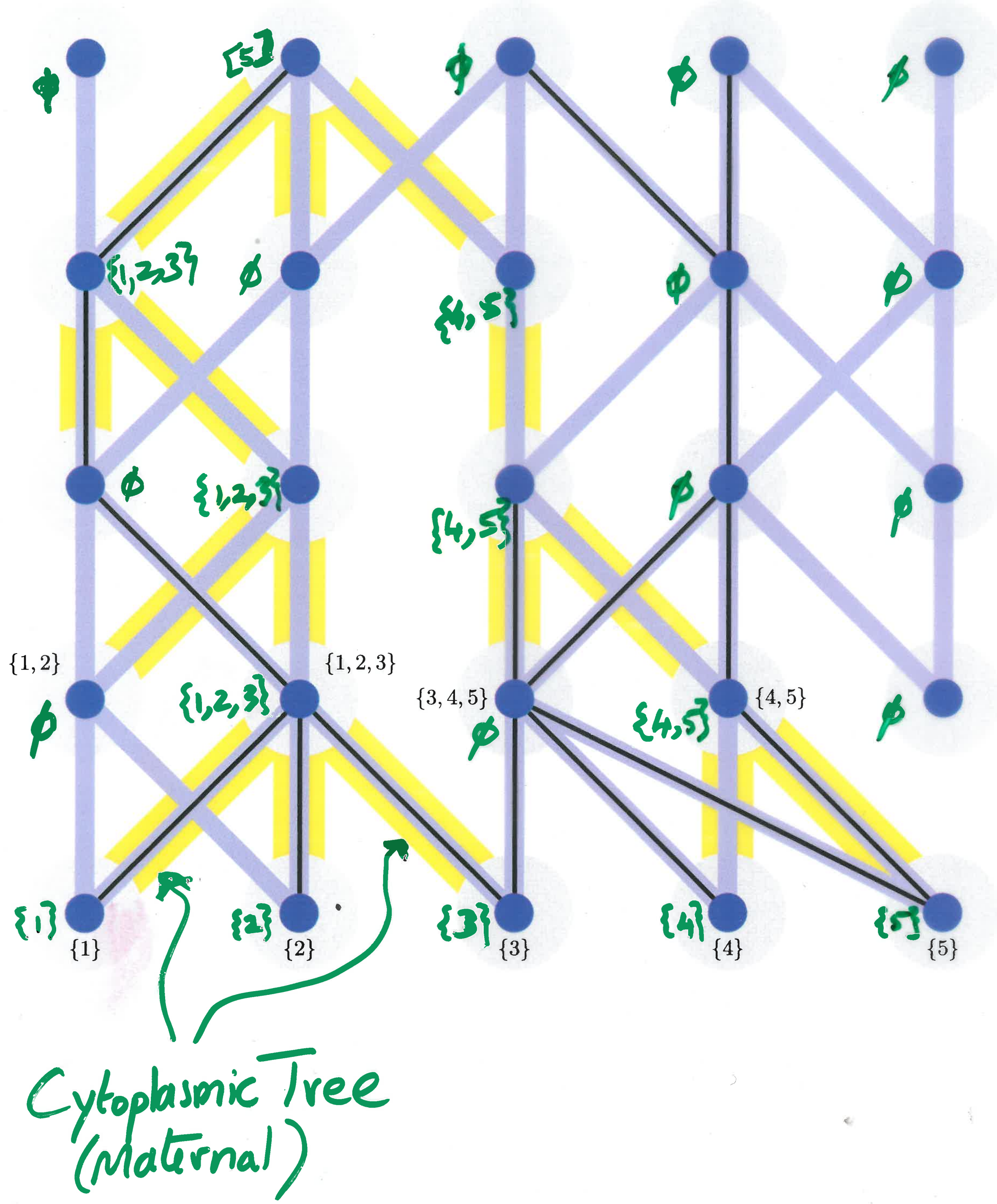
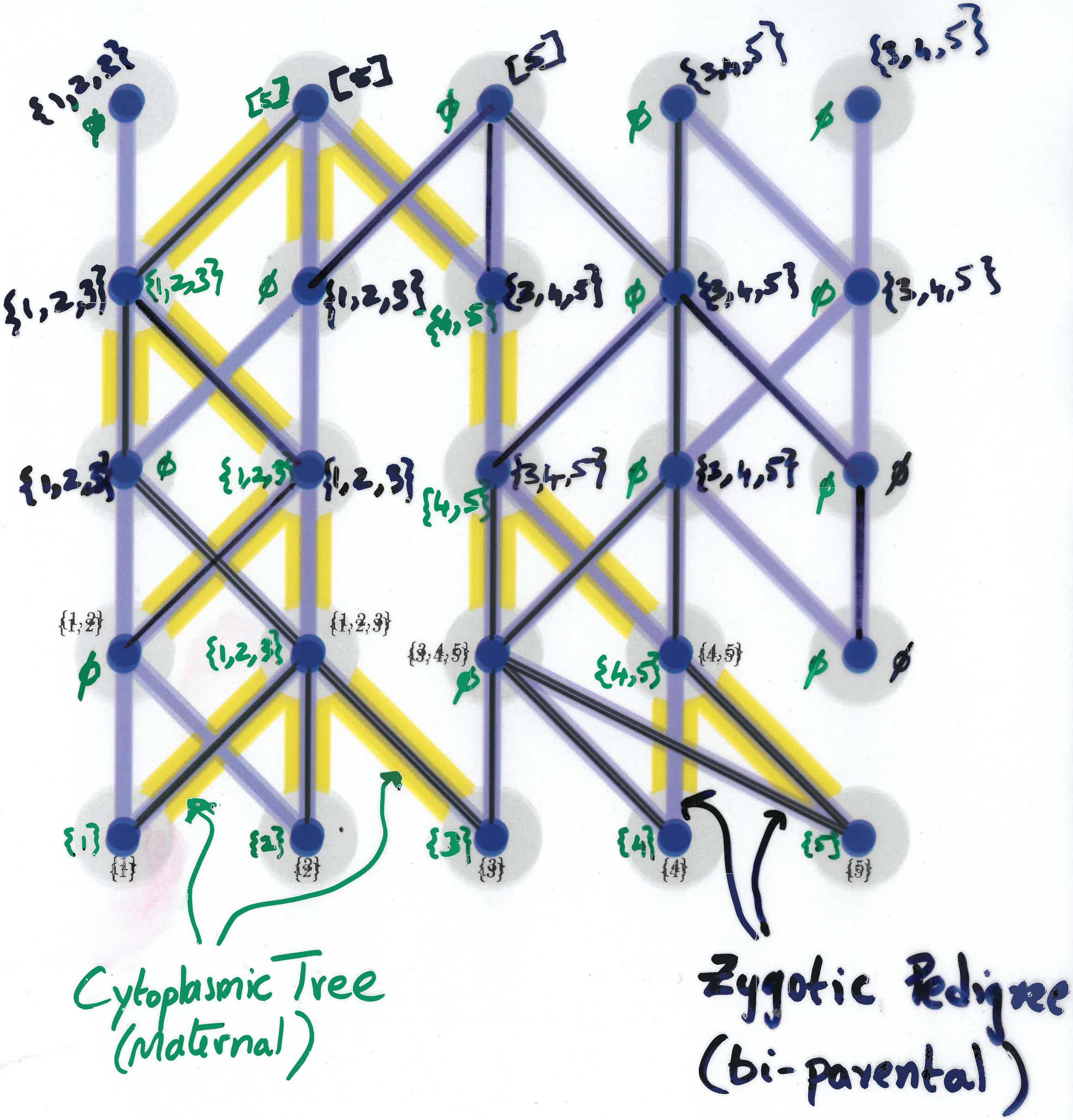

{1, 2}
{1, 2, 3}
{3, 4, 5}
{4, 5}
{1}
{2}
{3}
{4}
{5}
Cytoplasmic Tree
(Maternal)


{1,2}
{1,2,3}
{3,4,5}
{4,5}
{1}
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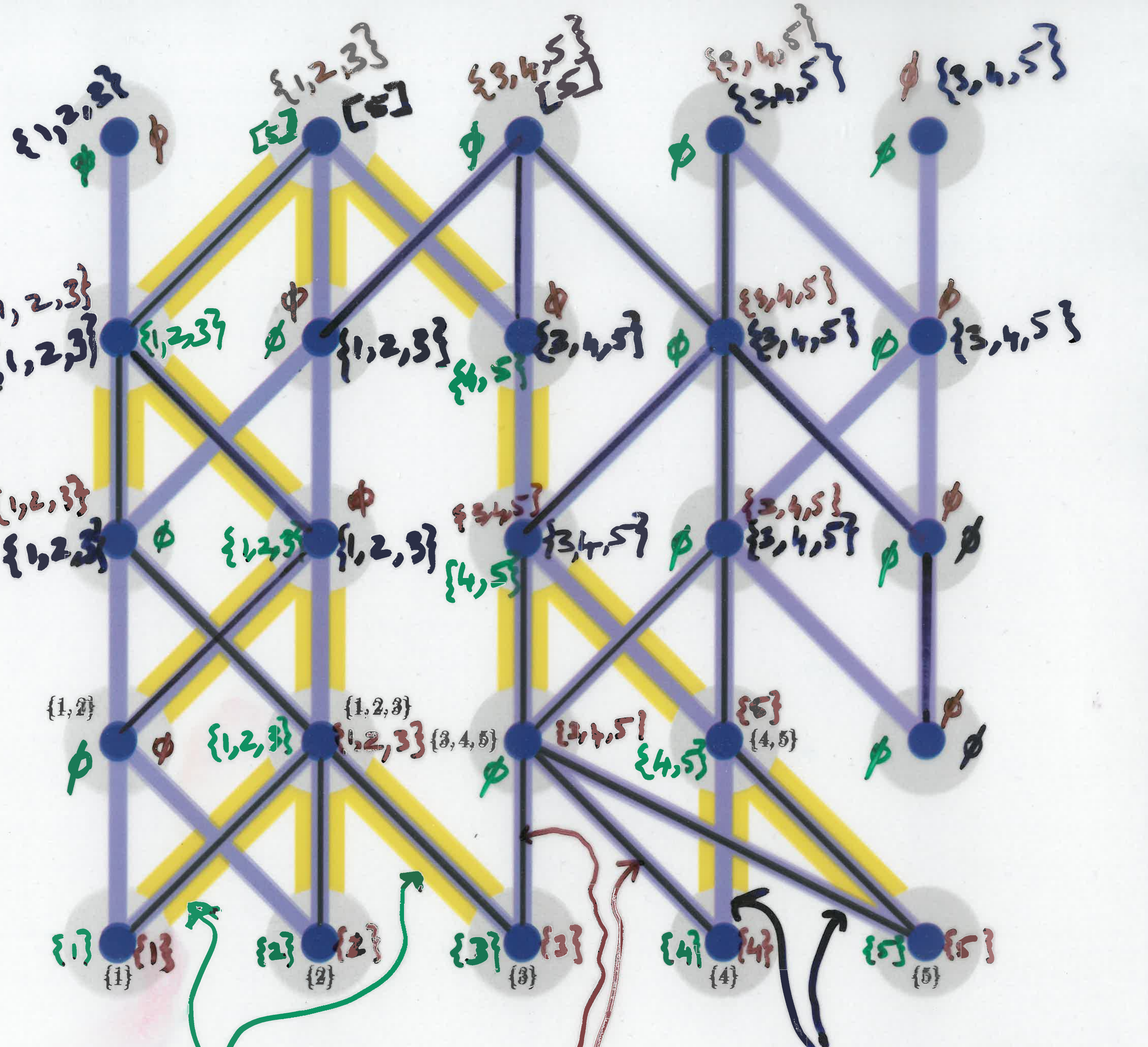
{5}
Cytoplasmic Tree
(Maternal)
Zygotic Pedigree
(bi-parental)



Cytoplasmic Tree
(Maternal)

$r = 0$

Sub-karyotic
Pedigree

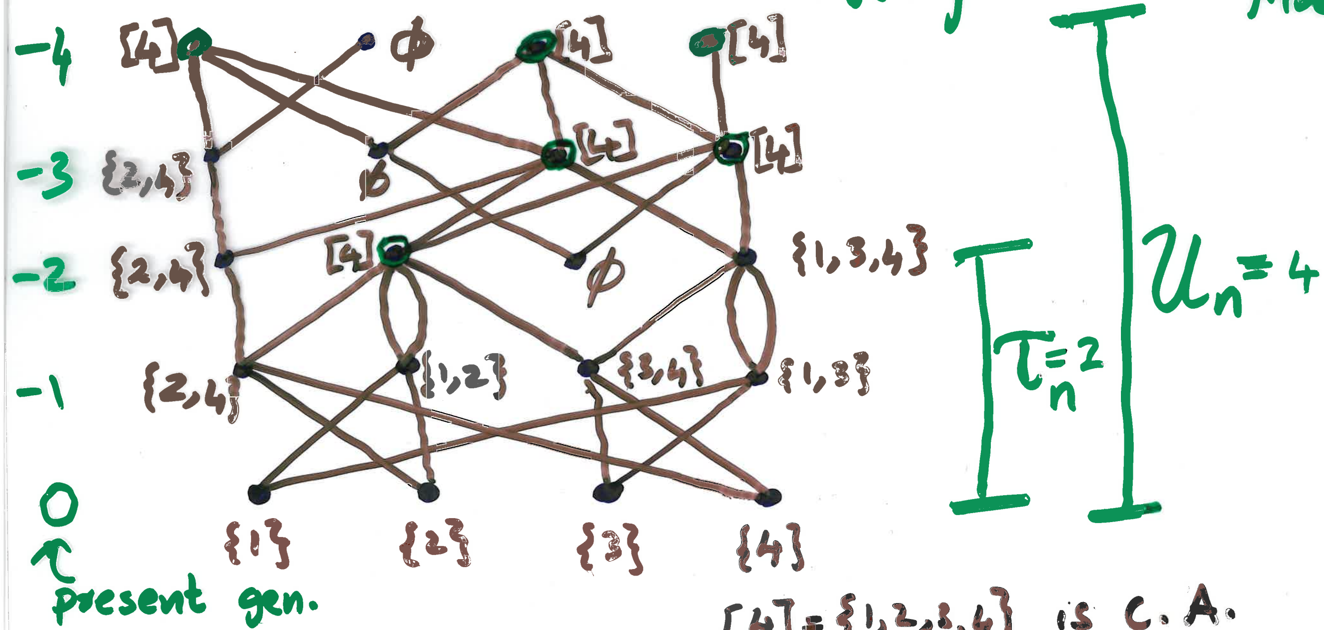
$0 < r < 1$

Zygotic Pedigree
(bi-parental)

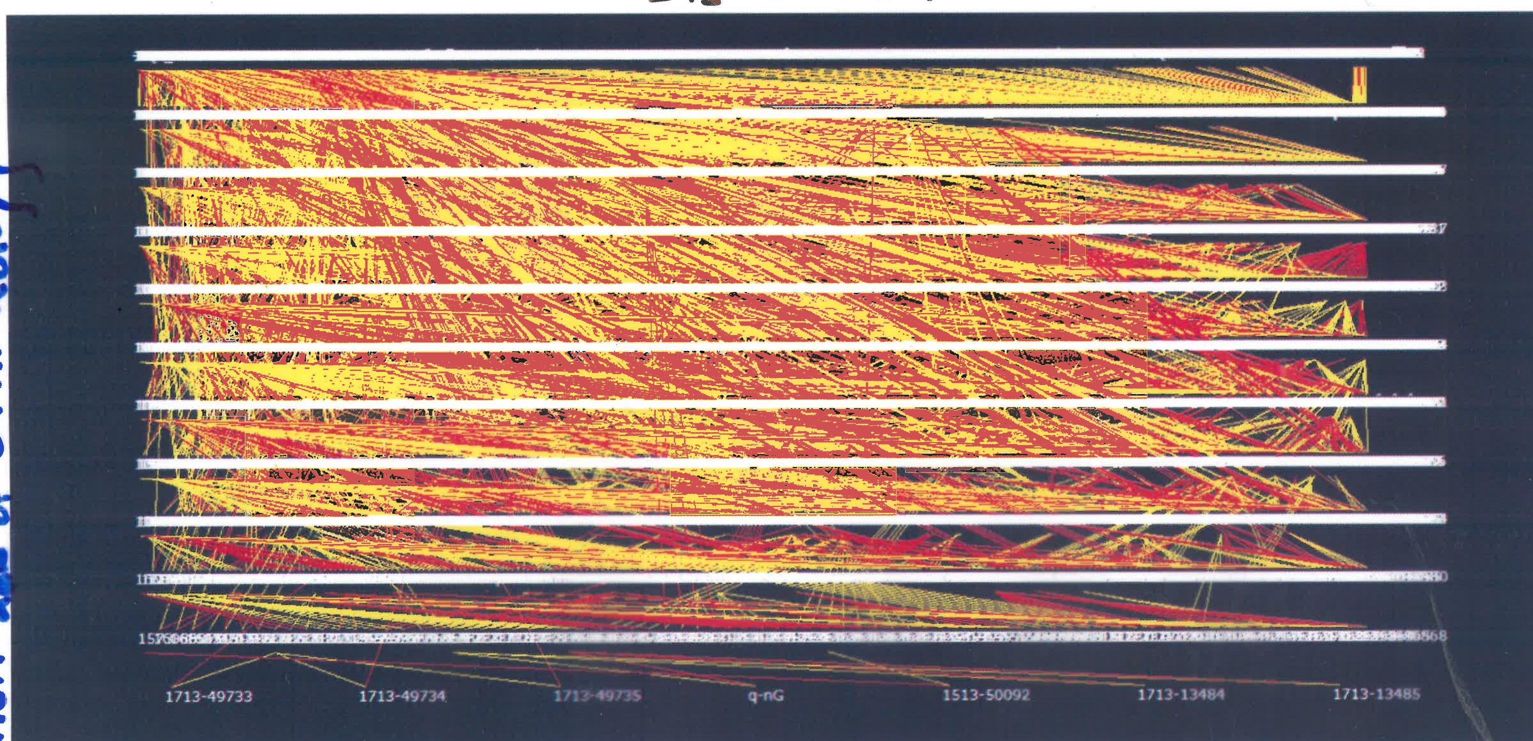
$r = 1$

Common Prob. Space from a Markov chain on Multi-set Partitions.

Hermaphroditic Wright Fisher (Chang's Model)



$[4] = \{1,2,3,4\}$ is C.A.

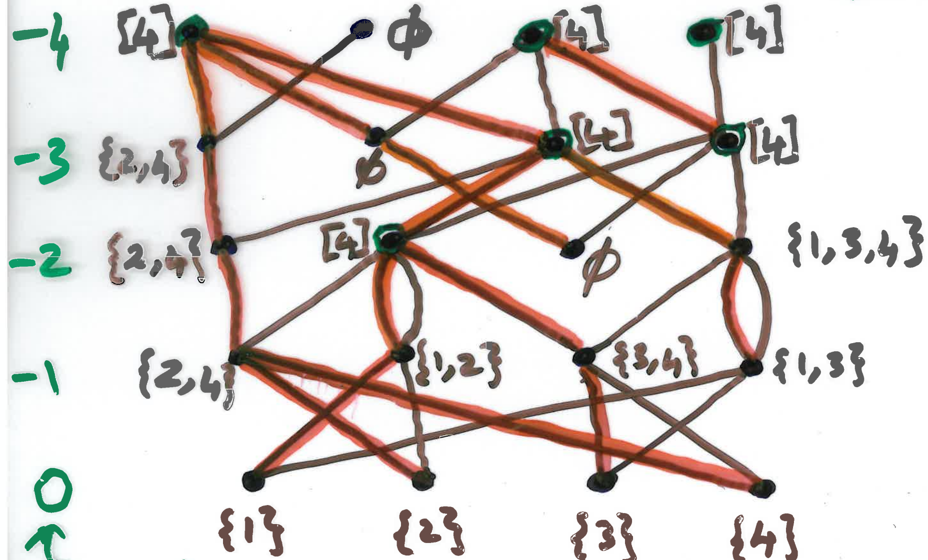


Nancy Chen (Clark Lab)

Cornell Lab of Ornithology

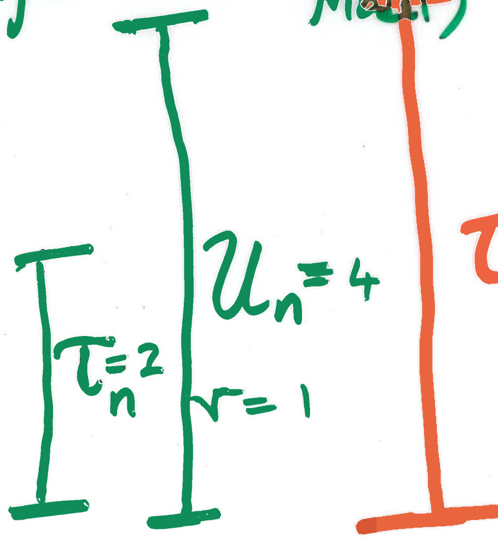
48 gens. Pedigree (10 gens DNA seqn.) of FL Scrub Jay

Hermaphroditic Wright Fisher (Chang's Model)



↑ present gen.

[4] = {1,2,3,4} is C.A.

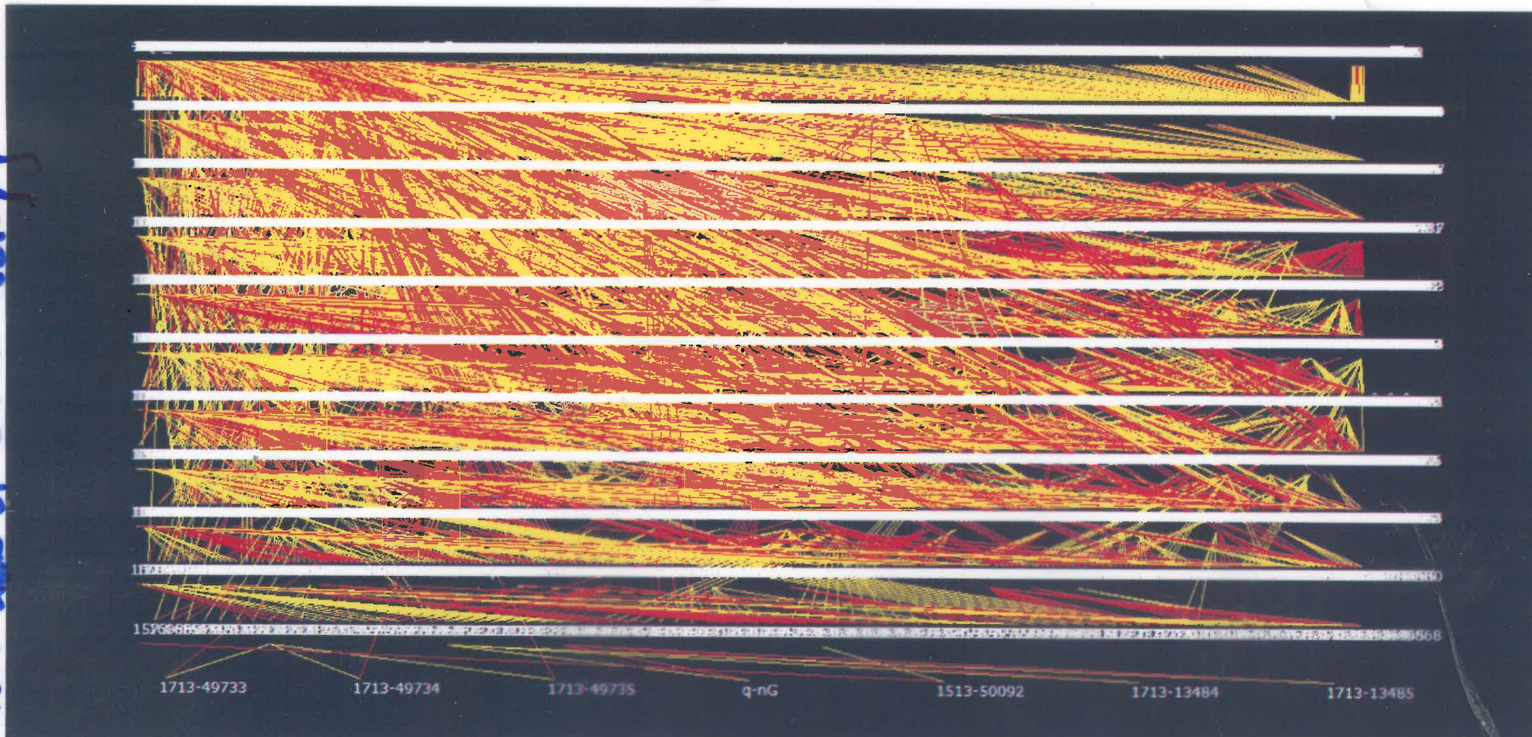


Haploid Wright-Fisher

$\downarrow n \to \infty$

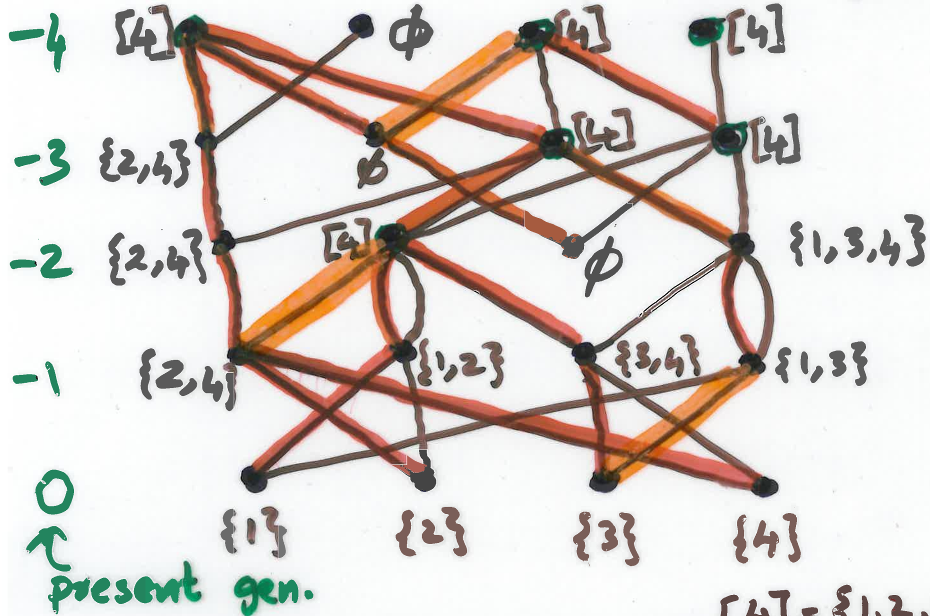
Kingman's Coalescent

Cornell Lab of Ornithology



Nancy Chen (Clark Lab)

48 gens. Pedigree (10 gens DNA segn.) of FL Scrub Jay



$[4] = \{1,2,3,4\}$ is C. A.

Hermaphroditic Wright Fisher (Chang's Model) $\xrightarrow{n\to\infty}$? $\tau_n, \mathcal{U}_n$

$\tau_n = 2$

$\mathcal{U}_n = 4$

$r = 1$

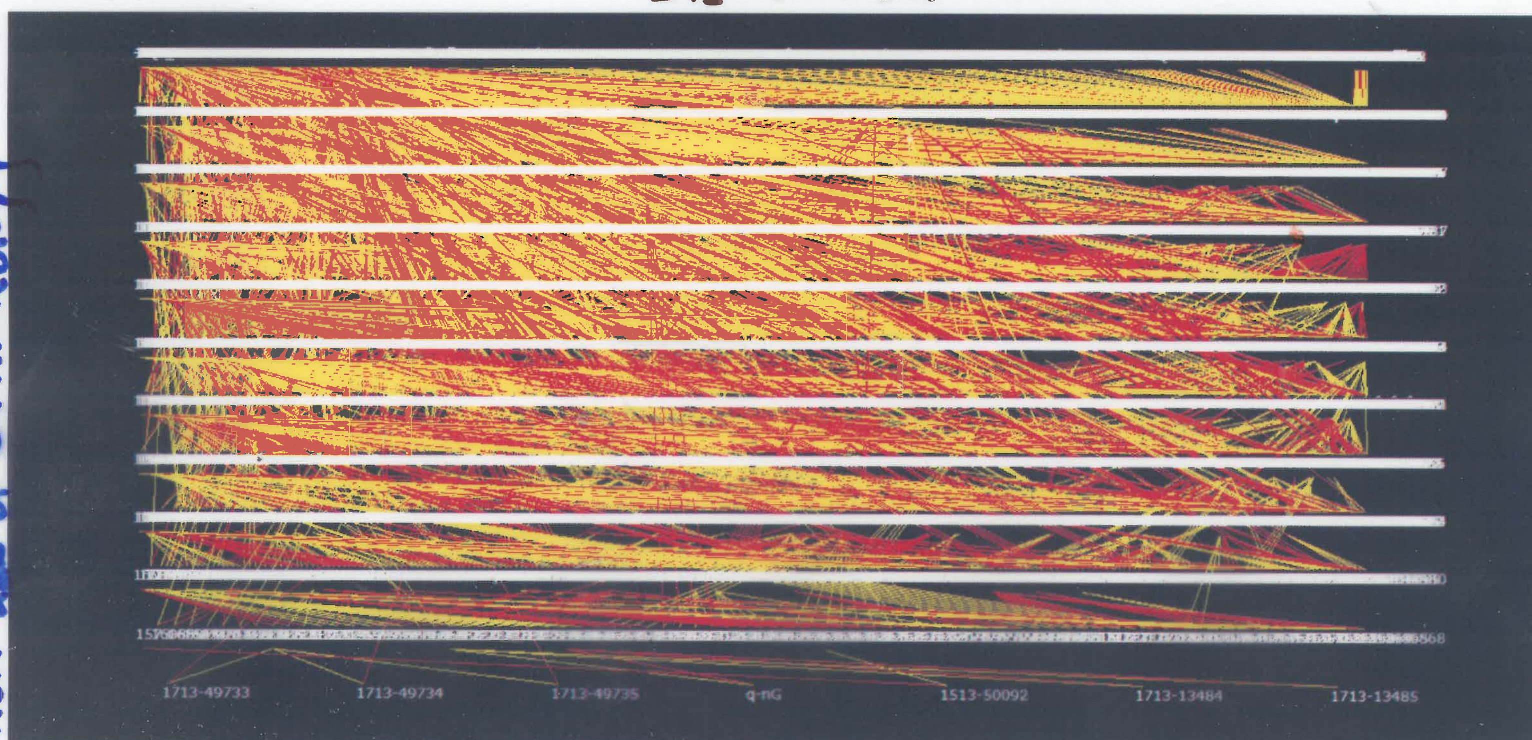
$\tau_n = \mathcal{U}_n = 4$

$r = 0$

Haploid Wright-Fisher

$\Big\downarrow n \to \infty$

Kingman's Coalescent



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48 gens. Pedigree (10 gens DNA seqn.) of FL Scrub Jay

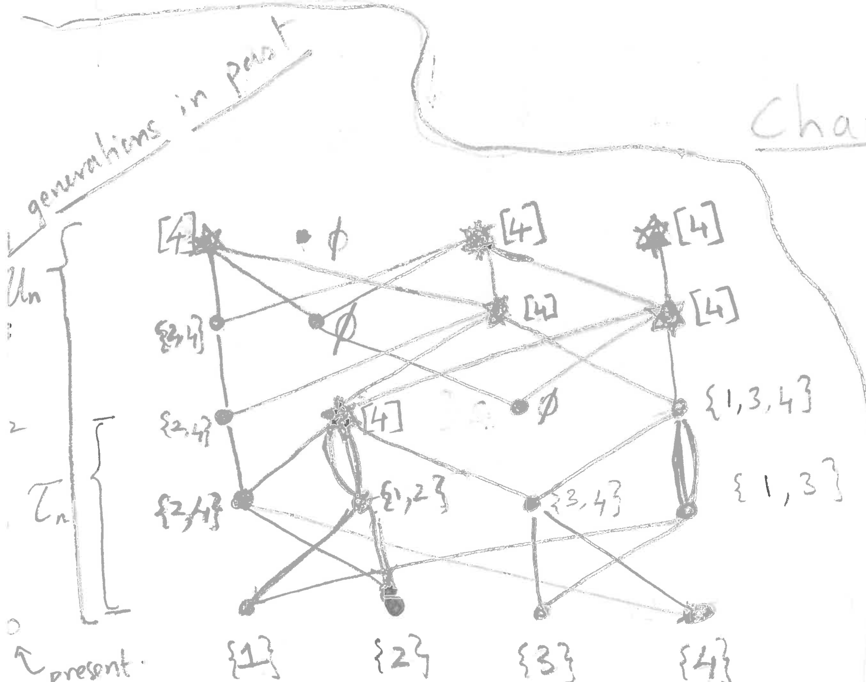
What if $0 < r < 1$?

Recombining Wright-Fisher

$\Big\downarrow n \to \infty$

? ${}^{r}\tau_n, {}^{r}\mathcal{U}_n$

consider a pop of constant size n (1) (2n haploid)



Chang's Model (Hermaphroditic W-F. process) (pop. pedigree).
Adv. Appl. Prob. 1999

$n = 4$

$r = 1 = Pr$ (choosing 2 parents in each generation)

$1-r = 0 = Pr$ (choosing 1 parent).

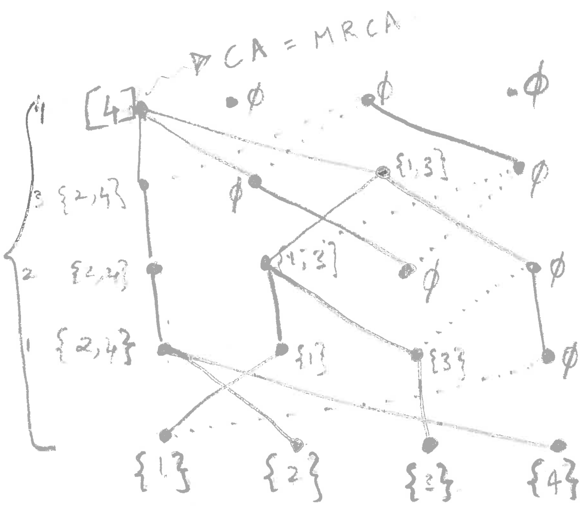
Thm 1: $\frac{\tau_n}{\log_2 n} \xrightarrow{P} 1$ as $n \to \infty$

Thm 2: $\frac{U_n}{(1+\zeta)\log_2 n} \xrightarrow{P} 1$ as $n \to \infty$

$\zeta = \frac{-1}{\log_2(\gamma)} \approx 0.7698$, γ is the smaller of the two solns. of $\gamma e^{-\gamma} = 2e^{-2}$

CA = ✡

extinct = ϕ



Wright-Fisher (haploid → single parent)

$r = 0 = Pr$ (choosing 2 parents)

$1-r = 1 = Pr$ (choosing 1 parent).

$E(\tau_n) = 2n$

$V(\tau_n) \approx O(n)$

with high Prob. $\tau_n \in [n, 4n]$

Show Transparencies:

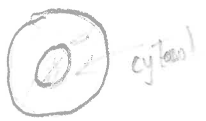
A natural Question: recombining Pop. pedigree (recombining (W-F) Hermaphroditic process)

what if $r \in (0,1) = Pr$ (choosing 2 parents).

$1-r = Pr$ (choosing 1 parent)

(Embedding a sequence of sub-graphs into $r=1$ pedigree)

Biol. Motivation:



- cytosolic (maternal tree) w/ paternal leakage prob. r
- sub-karyotic autosomal process of a recombining locus with per gen. recombination prob r.
- small popn! large n approx. ARGs

Now, does $\frac{{}^{r}\tau_n}{?} \xrightarrow{P} 1$

or $\frac{{}^{r}U_n}{} \xrightarrow{P} 1$

Wiuf-Hein Conjecture?

Nope! → ${}^{r}\tau_n$ is about $\log_{1+r}(n)$

since $E(\#$ parents$) = 1+r$.

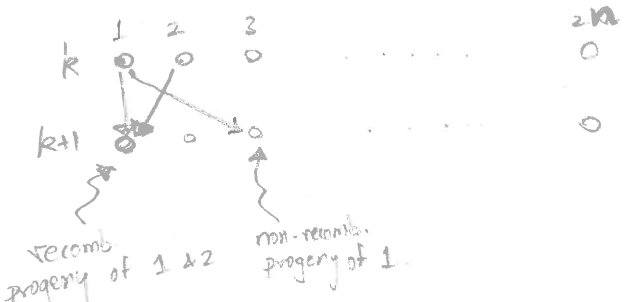
say first: Let's ask (not done before)

What is The discrete combinatorial structure underpinning the recombining Wright-Fisher Model. (forwards & backwards in time) → Markov chain for # of ancestors of a sample? (*) Markov lumping — The coarsest interesting resoln. of The random-graph-process.

Write: **Recombining W-F Model** (forward in time)

Let pop. size be $2n$ at every generation.

Let Label Set be $[2n] := \{1, 2, \ldots, 2n\}$ at gen. k



$V_i := \#$ of non-recombinant offspring of indiv. i (of gen. k)

$U_{i,j} := \#$ of recombinant " " pair of indivs. i,j $(i<j)$ (of gen k)

Totals:

$$V_\bullet = \sum_{i=1}^{2n} V_i \qquad U_{\bullet,\bullet} = \sum_{\{i,j \in [2n] : i<j\}} U_{i,j}$$

constant pop. size $\Rightarrow V_\bullet + U_{\bullet,\bullet} = 2n$.

Then

$$(V, U) = (\underbrace{V_1, V_2, \ldots, V_{2n}}_{2n}, \underbrace{U_{1,2}, U_{1,3}, \ldots, U_{2n-1,2n}}_{\binom{2n}{2}})$$

is Multinomial:

$$\Pr\big((V,U)=(v,u)\big) = \frac{(2n)!}{v_1! \cdots v_{2n}!\, u_{1,2}! \cdots u_{2n-1,2n}!}\, r^{u_{\bullet,\bullet}} \binom{2n}{2}^{-u_{\bullet,\bullet}} (1-r)^{v_\bullet} (2n)^{-v_\bullet}$$

This reproduction scheme is enforced iid in each gen.

Special case $(r=0)$ (non-recombin) standard W-F Model.

$V_\bullet = 2n$, $U_{\bullet,\bullet} = 0$ and $\Pr\big((V,U)=(v,0,\ldots,0)\big) = \dfrac{2n!}{v_1! \cdots v_{2n}!} \left(\dfrac{1}{2n}\right)^{2n}$

Special case $(r=1)$ Chang's (Zygotic) pedigree

$V_\bullet = 0$, $U_{\bullet,\bullet} = 2n$ and $\Pr\big((V,U)=(0,0,\ldots,0,u_{12},\ldots,u_{2n-1,2n})\big) = \dfrac{(2n)!\,\binom{2n}{2}^{-2n}}{u_{1,2}! \cdots u_{2n-1,2n}!}$

③

Now Ancestral size Markov Chain (backwards in time)
never done before explicitly

→ # of diploid indivs.
→ per gen prob. of recomb
→ # gens. ago

$$\left\{ {}^{2n,r}X(t) \right\}_{t \in \mathbb{Z}_-}$$ ← Fundamental Lumped Markov Chain

$\{0,-1,-2,\dots\}$

- discrete ARG models
  - 2 loci
  - unit intervals
    - Hudson's (sample only)
    - Griffiths' generalization

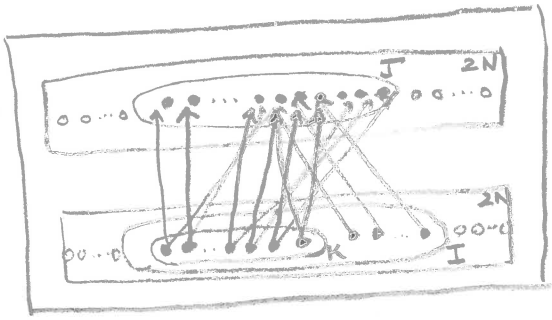
$$\mathbb{X} := \{1,2,\dots,2n\}$$

Thm 1
The One-step transition prob matrix of $\left\{ {}^{2n,r}X(t) \right\}$ is:

$$ {}^{2n,r}P_{i,j} = \binom{2n}{j} \sum_{k=0}^{i} \binom{i}{k} r^k (1-r)^{i-k} \frac{\sum_{m=0}^{j} \left[(-1)^m \binom{j}{m} \frac{(j-m)^i (j-m-1)^k}{2^k} \right]}{(2n)^{i-k} \binom{2n}{2}^k} \quad (*)$$

inclusion-exclusion **

Proof:
Fix sets I, J, K, s.t. $|J| = j$, $|I| = i$, $|K| = k$, $K \subseteq I$ ← recombinants; $I \setminus K$ ← non-recombinants



$$P(j \mid I) = \sum_{J : |J| = j} P(J \mid I) = \binom{2n}{j} \boxed{P(J \mid I)} \; (*)$$

$$P(J \mid I) = \sum_{K \subseteq I} r^{|K|} (1-r)^{|I|-|K|} \underbrace{P(J \mid I, K)}_{= \frac{|B(J \mid I,K)|}{(2n)^{|I|-|K|} \binom{2n}{2}^{|K|}}} \quad **$$

set of bipartite graphs with vertices $I \cup J$, with bipartition $J | I$, s.t.
vertices in K are of degree 2,
" " $I \setminus K$ " " 1
" " J are not isolated

→ # of ways in which indivs in I choose parent(s) so that K are recombin and $I \setminus K$ are not

all together: we have proved Thm 1.

$$P_{i,j} = P(j \mid i) = \binom{2n}{j} \sum_{k=0}^{i} \binom{i}{k} r^k (1-r)^{i-k} \frac{|B(j \mid i,k)|}{(2n)^{i-k} \binom{2n}{2}^k}$$

Thm 2 Let τ_n = # of gens. to MRCA of all present-day indivs. at a recombining locus (per gen. recomb. prob. r) in diploid popn. of size n.

Then $\forall\ \varepsilon > 0$,

$$\lim_{n\to\infty} P\left\{ (1-\varepsilon)\, C(r) \ln n \le \tau_n \le (1+\varepsilon)\, C(r) \ln n \right\} = 1.$$

$$C(r) = \frac{1}{\ln(1+r)} \boxed{- \frac{1}{\ln(1-r)}}$$

because it takes more time to reach n (unlike Chang's case with $r=1$)

this term is missing in Wiuf-Hien conjecture.

Thm 3 Let, U_n = # of gens. ago when each indiv. is either a CA or not a CA of all present-day indivs.

$\rho = \rho(r)$ be unique soln. in $(0,1)$ to $x = e^{-(1+r)(1-x)}$

i.e. Prob $\{$extinction of GW(Pois(1+r))$\}$

Then, $\forall\ \varepsilon > 0$, $\lim_{n\to\infty} P\Big\{ (1-\varepsilon)\Big(C(r) - \frac{1}{\ln(\rho(1+r))}\Big) \ln n$

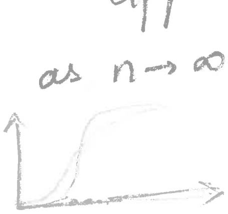
waiting for all families to have 0 or $(\ln n)^2$ descendants — extra time is due to

$\le U_n$

$\le (1+\varepsilon)\Big(C(r) - \frac{1}{\ln(\rho(1+r))} - \frac{1}{\ln(1-r)}\Big) \ln n \Big\} = 1$

Corr. 4 The fraction of popn at U_n that are CAs is $\boxed{1-\rho(r)}$.

Thm 5. $\underbrace{\text{Griffiths ARG}}_{\{q(i,i-1) = \binom{i}{2},\ q(i,i+1) = \rho i\}}$ (Hudson) is the asymptotic approx. of $\{ {}^{2n,r}X_t \}$, based on a random sample as $n \to \infty$ but nr is held constant.



(8)

$n = 10^4,\ r = 0.1$

Reps	T_n	U_n	CAs/n
1	117	223	0.17
2	120	232	0.18
3	129	228	0.18
4	123	215	0.18
5	121	210	0.17
	184	[278, 365]	0.176
	↑ Jeerakan Thm 2	↑ Thm 3	↑ 1-p Cor. 4.

$n = 10^4,\ r = 0.5$

T_n	U_n	CAs/n
29	60	0.58
30	53	0.58
29	61	0.58
29	51	0.58
29	52	0.58
36	[55, 68]	0.58

$n = 10^4,\ r = 0.9$

T_n	U_n	CAs/n
16	30	0.76
16	32	0.76
16	30	0.77
16	31	0.77
16	30	0.77
18	[30, 34]	0.77.

Note: if $r = 0.01$ then

[820, 930]	[1560, 1800]	0.02
1842	[2764, 3680]	0.02

so T_n increases as r decrea[ses]

Also $\Pi(i/n) = \lim_{t\to\infty} X_t^{n,r}(i)/n \xrightarrow[n\to\infty]{} 1 - p(r)$

Nowhere near asymptotic regime .... (proof needed).

Now Consider a Natural Embedding (a ~~consecutive~~ thinning scheme)

$\boxed{\text{Ancestry levels}}$

Level 0 $\quad r = 1$ (all genomic content)

Let ω be a sample pedigree ($r=1$)

Level 1

consider a large fragment of recomb. DNA, say chr 1. with $r = 0.9$

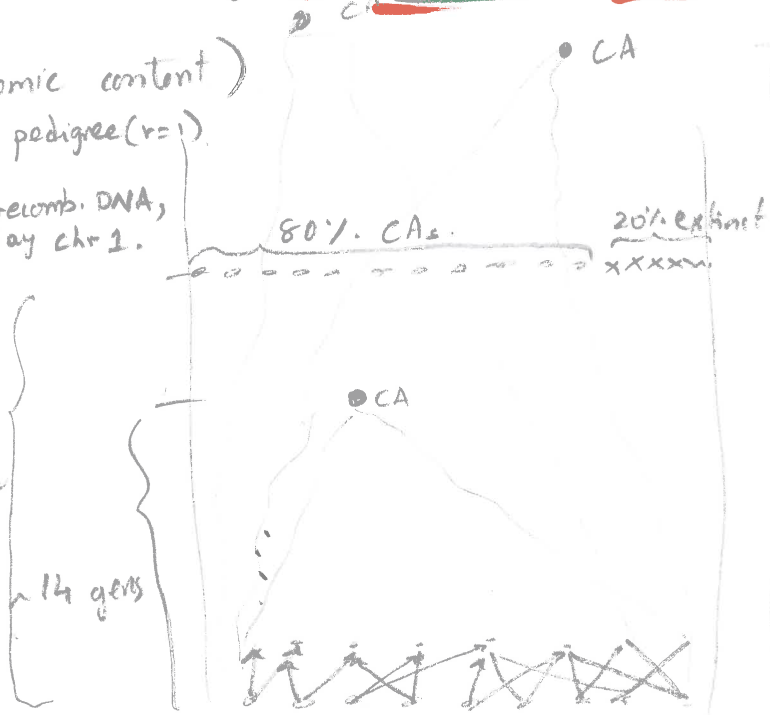
Get ω' from ω by thinning.

Level 2

consider a sub-fragment with $r = 0.5$

Get ω'' from ω' ~30 gens

by further thinning.

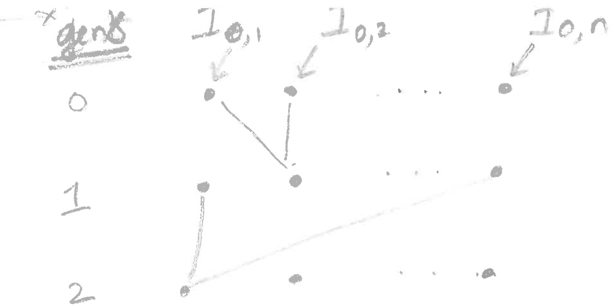


T_n for small r

⋮

Level 1′ consider a different large fragment, say chr. 2

Level 2′ consider its sub-fragment ...



$\mathcal{G}_t^i$ = Set of descendants of $I_{0,i}$ after t generations.

$$G_t^i = \left|\mathcal{G}_t^i\right| = \boxed{\#\text{ of descendants of } I = I_{0,i}} \quad (*)$$

some indiv in gen. 0

$\{G_t^i\}_{t\in\mathbb{Z}_+}$ is a M.C. with tr. pr.

$$\left(G_{t+1}^i \,\middle|\, G_t^i\right) \sim \mathrm{Bin}\Bigg(n, \underbrace{\overbrace{\boxed{(1+r)\frac{G_t^i}{n}}}^{\text{super-critical}} - r\,\frac{(G_t^i)^2}{n^2}}_{=\,(1-r)\frac{G_t^i}{n} + r\left(1-\left(1-\frac{G_t^i}{n}\right)^2\right)}\Bigg)$$

1. Stage G1 → (*) reaches at least $(\ln n)^2$ after $\tau_n^{(G1)} \approx 2\ln\ln n/\ln(1+r)$ gens. with high prob.

2. Stage G2

   (*) increases from $(\ln n)^2$ to $g_2 n$ in $\tau_n^{(G2)} \approx \frac{\ln n}{\ln(1+r)}$

   $g_2 \in (0, 1/2)$ depending on ε

3. Stage G3 → (*) increases from $g_2 n$ to $n/2$ in $\tau_n^{(G3)}$ gens,

   s.t. $P\left(\tau_n^{(G3)} \le \ln\ln n\right) = 1 - o(1/n)$

Let $B_t^i = n - G_t^i$ be $\boxed{\#\text{ of non-descendants of indiv. } I = I_{0,i}}$ (**) after t gens

$\{B_t^i\}_{t\in\mathbb{Z}_+}$ is a M.C. with tr. pr.

$$\left(B_{t+1}^i \,\middle|\, B_t^i\right) \sim \mathrm{Bin}\Bigg(n, \overbrace{\boxed{(1-r)\frac{B_t^i}{n}}}^{\text{sub-critical}} + r\,\frac{(B_t^i)^2}{n^2}\Bigg).$$

4. Stage B1 → (**) decreases from $n/2$ to $b_1 n$ ($\in (0, 1/2)$)

   in $\tau_n^{(B1)}$ gens, $P\left(\tau_n^{(B1)} \le \ln\ln n\right) = 1 - o(1/n)$

5. Stage B2 (**) further decrease from $b_1 n$ to $(\ln n)^2$ in $\tau_n^{(B2)} \approx \frac{-\ln n}{\ln(1-r)}$ gens

6. Stage B3 (**) go extinct from $(\ln n)^2$ to 0 in $\tau_n^{(B3)} \approx \frac{-2\ln\ln n}{\ln(1-r)}$ gens.

All 6 stages together gives. with prob $\to 1$ as $n \to \infty$. (7)

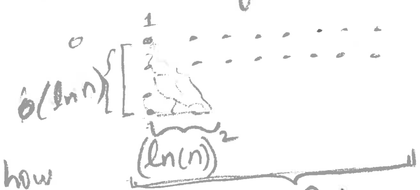
The first time when an indiv. becomes a C.A.

$\longrightarrow x$ ———

Define.

Proof   stage G1

Let $\{X_t\}_{t \in \mathbb{Z}_+}$ be some $\mathbb{Z}_+$-valued process. $\tau_s^X = \inf\{t : X_t = s\}$



Show with High Prob. within $o(\ln n)$ gens. There will be at least $(\ln(n))^2$ descendants of an indiv. from $t=0$.

Idea: Approx. "# of descendants of I process" $\{G_t\}_t$ using $\{Y_t^+\}_t \sim GW(\text{Pois}(1+r))$

provided we look early enough & with popn. size small enough.

Lemma 17(i)(a). If $k_n \to \infty$, $b_n \to \infty$ as $n \to \infty$ s.t. $k_n b_n^2 = o(n)$

Then $\mathbb{P}_1\left[\tau_{b_n}^{G} > k_n\right] = \mathbb{P}_1\left[\tau_{b_n}^{Y^+} > k_n\right](1+o(1))$ ✡

Starting at 1, $G_0 = 1$

Proof: Bounding the ratio of transition probs. of G & Y^+ we can get

$$P_1\{\tau_{b_n}^{Y} > k_n\} e^{-c k_n b_n^2/n} \le P_1\{\tau_{b_n}^{G} > k_n\} \le P_1\{\tau_{b_n}^{Y^+} > k_n\} e^{c k_n b_n^2/n}.$$

so, need $k_n b_n^2 = o(n)$. for ✡ to hold.

Take $b_n = (\ln n)^2$ $j_n = \frac{3}{\ln(1+r)} \ln\ln n$, $m_n = \ln\ln n$ many geometric trials

j_n: fails $< b_n$ w.p. $1-c$

Prob. failure in m_n geom. trials is $(1-c)^{m_n} \to 0$.

success in one geometric trial.

By ✡ & a non-neg. Martingale argument:

$$\liminf_{n\to\infty} P\left\{\tau_{(\ln n)^2} \le \frac{3}{\ln(1+r)} \ln\ln n\right\} = c > 0$$

$\{M_t\} = \{Y_t^+ (1+r)^{-t}\}$ is a non-neg. mart $\xrightarrow{a.s.} M_\infty$, $P\{M_\infty = 0\} = p(r) < 1$

Since:

$$\limsup P\left\{\tau_{(\ln n)^2}^{G} > \frac{3\ln\ln n}{\ln(1+r)}\right\} \overset{\text{By ✡}}{\le} \limsup P\left\{\tau_{(\ln n)^2}^{Y^+} > \frac{3\ln\ln n}{\ln(1+r)}\right\}$$

$$\le \limsup P\Big\{M_{\frac{3\ln\ln n}{\ln(1+r)}} < \underbrace{(\ln n)^2 (1+r)^{-\frac{3\ln\ln n}{\ln(1+r)}}}_{1/\ln n}\Big\} \overset{\text{Fatou's Lemma}}{\le} P\left\{\limsup\{\cdot\}\right\}$$

$$= P\{M_{3\ln\ln n/\ln(1+r)} < 1/\ln n \text{ i.o.}\} \le P\{M_\infty = 0\} = p(r) < 1.$$

So, $\liminf P\left\{\tau_{(\ln n)^2}^{G} \le \frac{3\ln\ln n}{\ln(1+r)}\right\} \ge 1 - p(r) > 0$.

Proof of Stage G1 complete:

Thus, $Pr\left\{\begin{array}{l}\text{no indiv. at time 0 has at least } (\ln n)^2 \\ \text{descends. after } (\ln\ln n)\dfrac{3\ln\ln n}{\ln(1+r)} = o(\ln n) \text{ gens}\end{array}\right\}$

$$< (1-c)^{\ln\ln n} \longrightarrow 0 \quad \text{as } n \to \infty.$$

$\boxed{\text{i.e. } \lim_{n\to\infty} P\left\{\tau_n^{(G1)} > \frac{3}{\ln(1+r)}(\ln\ln n)^2\right\} = 1}$

Next we study The family size of such a thriving indiv I from gen. 0 with $\geq (\ln n)^2$ descends. by time $o(\ln n)$.

Pf Stage G2
$$G_t \geq (\ln n)^2 \longrightarrow G_t \geq g_2 n \quad , \; g_2 \in (0, \tfrac{1}{2})$$

Idea. G_t is large enough to behave like its expectation.

Use Bernstein's $\neq$. If $X \sim \text{Bin}(n,p)$, $x > 0$, Then

$$P\{X \geq np + x\} \leq \exp\left(\frac{-x^2}{2np(1-p) + (2/3)x}\right)$$

and
$$P\{X \leq np - x\} \leq \quad ''$$

With $x = \eta G_t - r G_t^2/n > 0$, $\eta > 0$ s.t. $\underbrace{\ln(1+r-\eta) > \frac{\ln(1+r)}{1+\varepsilon/2}}_{\text{i}}$, $g_2 <$ ... (ii)

in $G_{t+1} \mid G_t \sim \text{Bin}(n, \underbrace{(1+r)G_t/n - r(G_t/n)^2}_{p})$ & Bernst $\neq$:

$$\boxed{P\{G_{t+1} < \overbrace{(1+r-\eta)G_t}^{np-x}, (\ln n)^2 \leq G_t \leq g_2 n\}} \leq \exp\left(-\frac{(\overbrace{\eta - r g_2}^{>0 \text{ (ii)}})^2 (\ln n)^2}{2(1+r) + 2\eta/3}\right) \to 0$$

If $G_{t+1} \geq (1+r-\eta)G_t \quad \forall\, t \leq m_n := \left\lceil \frac{\ln n - 2\ln\ln n + \ln g_2}{\ln(1+r-\eta)} \right\rceil$

Then $G_{m_n} \geq (1+r-\eta)^{m_n} G_0 \geq g_2 n$ is satisfied.

Thus,
$$P\{G_t < g_2 n, \forall t \leq m_n \mid G_0 \geq (\ln n)^2\} \leq \sum_{t=0}^{m_n} P\{\blacktriangle\} \leq m_n e^{-c(r,\eta,g_2)(\ln n)^2} = o(\tfrac{1}{n})$$

Let $\tau_n^{(G2)} = \inf\{t : G_t \geq g_2 n\}$ and $G_0 \geq (\ln n)^2$

Then
$$P\left\{\tau_n^{(G2)} > \overbrace{\left(1+\frac{\varepsilon}{2}\right)\frac{\ln n}{\ln(1+r)}}^{\text{by (i) \& } m_n \to \frac{\ln n}{\ln(1+r-\eta)}}\right\} = o\left(\frac{1}{n}\right)$$
as $n \to \infty$

IIIly, $(\ln n)^2 \leq G_0 \leq (\ln n)^3$ as $n \to \infty$ by other Bernst $\neq$

$$P\left\{\tau_n^{(G2)} < \left(1-\frac{\varepsilon}{2}\right)\frac{\ln n}{\ln(1+r)}\right\} = o\left(\frac{1}{n}\right)$$

Stage G2 done

Stage G3 From $G_t \geq g_2 n$ to $G_t \geq n/2$. Deterministic + Idiot.

same idea in pf of stage G2 gives:

Let $G_0 \geq g_2 n$ where $0 < g_2 < \frac{\eta}{r}$, $\ln(1+r-\eta) > \frac{\ln(1+r)}{1+\varepsilon/2}$.

$$\tau_n^{(G3)} = \inf\{t : G_t \geq n/2\}$$

Then as $n \to \infty$,

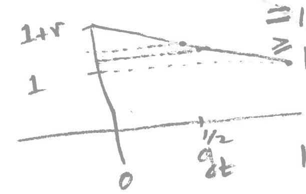
$$P\{\tau_n^{(G3)} > \ln\ln n\} = o\left(\frac{1}{n}\right)$$

Amandine's Lemma

Amandine's proof

Detailed Version

consider $g_t = \frac{G_t}{n}$, $E(g_{t+1} | g_t) = \left[n\left[\frac{(1+r)G_t}{n} - r\frac{G_t^2}{n^2}\right] \cdot \frac{1}{n}\right] = (1+r)g_t - rg_t^2 = g_t \overbrace{\underbrace{(1+r-rg_t)}_{\geq 1+r-\frac{r}{2} = 1+\frac{r}{2} \geq 1+\frac{r}{3}}}^{\text{mullip. factor}}$



So, need $g_2 n \left[1 + \frac{r}{3}\right]^{m_n} = \frac{1}{2} n$.

$$\therefore\ m_n = \log_{1+\frac{r}{3}}\left(\frac{n}{2} \cdot \frac{1}{g_2 n}\right) = \log_{1+\frac{r}{3}}\left(\frac{1}{2g_2}\right)$$

Let $G_0 \geq g_2 n$, $\tau_n^{(G3)} = \inf\{t : G_t \geq \frac{n}{2}\}$ as $n \to \infty$.

Then $P\left\{\tau_n^{(G3)} > \log_{1+\frac{r}{3}}\left(\frac{1}{2g_2}\right)\right\} = o\left(\frac{1}{n}\right)$

Pf:

For $g_2 n \leq G_t \leq n/2$, by Berns. $\neq$:

$$P\left\{G_{t+1} \leq \left(1+\frac{r}{3}\right)G_t \,\middle|\, G_t\right\} \leq \exp(-c(r)g_2 n)$$

So, if $\tau_n^{(G3)} > \log_{1+\frac{r}{3}}\left(\frac{1}{2g_2}\right)$ then there was some $t < \log_{1+\frac{r}{3}}\left(\frac{1}{2g_2}\right)$ s.t. $g_2 n \leq G_t \leq n/2$. Thus, $P\left\{\tau_n^{(G3)} > \log_{1+\frac{r}{3}}\left(\frac{1}{2g_2}\right)\right\} \leq \underbrace{\log_{1+\frac{r}{3}}\left(\frac{1}{2g_2}\right)e^{-c(r)g_2 n}}_{= o(1)}$ as $n \to \infty$

Chang's proof

Stage B1: from $B_t \le n/2$ to $B_t < b_1 n$, $b_1 \in (0, 1/2)$

$$B_t = n - G_t = \# \text{ of non-desc. of } I.$$

$$(B_{t+1} \mid B_t) \sim \text{Bin}\Big(n, \underbrace{(1-r)\frac{B_t}{n}}_{\text{sub-critical}} + r\frac{B_t^2}{n}\Big).$$

Here, B_t decreases nearly deterministically (being of order n) at a rate $\ge 1 - r + r/2 = 1 - r/2$.

(Exactly same arguments as in G3 phase) we get: $\tau_n^{(B1)} = \inf\{t : B_t \le b_1 n\}$

If $B_0 \le n/2$ and $b_1 \in (0, 1/2)$.
Then as $n \to \infty$ $P\{\tau_n^{(B1)} > \ln \ln n\} = o\left(\frac{1}{n}\right)$.

Stage B2: from $B_t \le b_1 n$ to $B_t \le (\ln n)^2$

Proceeding as in G2.
Let $\eta \in (0, 1-r)$ and fix $b_1 > 0$ s.t. $\eta > r b_1$, By Bernst $\neq$,

$$P\{B_{t+1} > (1-r+\eta) B_t, (\ln n)^2 \le B_t \le b_1 n\} \le \exp\left\{-\frac{(\eta - r b_1)^2 (\ln n)^2}{2 + 2\eta/3}\right\}.$$

Let $m_n' = \left\lceil \dfrac{\ln n - 2 \ln \ln n + \ln b_1}{-\ln(1-r+\eta)} \right\rceil$.

If $B_{t+1} \le (1-r+\eta) B_t$, $\forall\, t \le m_n'$

Then $G_{m_n'} \le (1-r+\eta)^{m_n'} B_0 \le (\ln n)^2$

so, $P\{B_t > (\ln n)^2, \forall t \le m_n' \mid B_0 \le b_1 n\} \le m_n' e^{-C'(r,\eta,b_1)(\ln n)^2} = o(1/n)$

and if we choose $\eta > 0$ and b_1 s.t. $\ln(1-r+\eta) \le \dfrac{\ln(1-r)}{1+\varepsilon/2}$ & $b_1 < \dfrac{\eta}{r}$

Then we get:

Lemma 11. If $B_0 \le b_1 n$, $\tau_n^{(B2)} = \inf\{t : B_t \le (\ln n)^2\}$.

Then as $n \to \infty$ $P\left\{\tau_n^{(B2)} > \left(1+\frac{\varepsilon}{2}\right)\frac{\ln n}{-\ln(1-r)}\right\} = o\left(\frac{1}{n}\right)$

Lemma 12 Likewise, (up to taking smaller η & b_1).

If $n/\log n \le B_0 \le b_1 n$. Then

as $n \to \infty$ $P\left\{\tau_n^{(B2)} < \left(1-\frac{\varepsilon}{2}\right)\frac{\ln n}{-\ln(1-r)}\right\} = o\left(\frac{1}{n}\right)$

Stage B3. Extinction of $\{B_t\}_{t \in \mathbb{Z}_+}$

Let $B_0 \le (\ln n)^2$, $\tau_n^{(B3)} = \inf\{t : B_t = 0\}$.

Lemma 17 (ii): If for some $\alpha \in (0, \frac{1}{4})$, $\gamma \in (2\alpha, \frac{1}{2})$ we have $i = O(n^\alpha)$ and $k = o(n^{1-2\gamma})$, then as $n \to \infty$

$$P_i\{\underbrace{\tau_0^B}_{=\inf\{t : B_t = 0\}} > k\} = P_i\{\tau_0^{Y^-} > k\}(1+o(1))$$

$Y^- \sim GW(\mathrm{Pois}(1-r))$.

with $\alpha = 0$, $\gamma = \frac{1}{4}$ in lemma 17 (ii)

$$P_{B_0}\{\tau_n^{(B3)} > c \ln\ln n\} = P_{B_0}\{\tau_0^{Y^-} > c \ln\ln n\}(1+o(1))$$

↳ first time Y^- goes extinct starting from B_0.

By lemma 15 & branching prop.

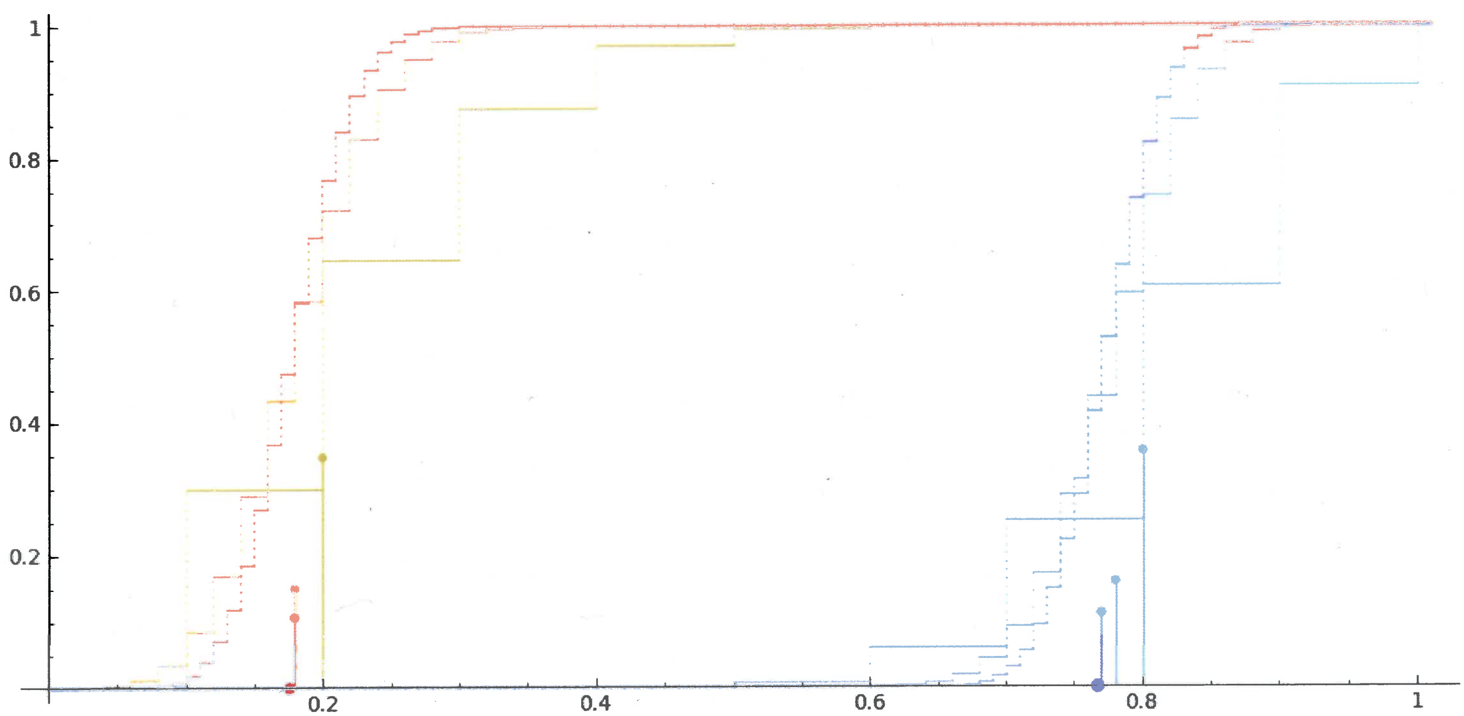
$\tau_0 = \inf\{t : Y_t = 0\}$
$Y_t \sim GW\left(\underset{<1}{\text{offsp. mean } m}\right)$
Then, $\forall k \in \mathbb{Z}_+$
$P_1\{\tau_0 > k\} < m^k$

$$P_{B_0}\{\tau_0^{Y^-} \le c \ln\ln n\} = \left(P_1\{\tau_0^{Y^-} \le c \ln\ln n\}\right)^{B_0}$$

$$\ge \left(1 - (1-r)^{c \ln\ln n}\right)^{(\ln n)^2}$$

$$\text{(By Taylor exp.)} = e^{-(\ln n)^2\left[(1-r)^{c\ln\ln n} + o\left((1-r)^{c\ln\ln n}\right)\right]}$$

$$\longrightarrow 1 \quad \text{as } n \to \infty \text{ if } c > \frac{-2}{\ln(1-r)}$$



1
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1