There and back again: split and prune to tighten
a tree arithmetic with **mapped regular pavings**

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Main Idea & Motivation
- Motivating Examples
- Why MRPs?

Theory of Regular Pavings (RPs)

Theory of Mapped Regular Pavings (MRPs)

Randomized Algorithms for $\mathbb{IR}$-MRPs

Applications of Mapped Regular Pavings (MRPs)

Conclusions and References

Context & Disclaimers

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2. **Engineering Problem**: Resource-limited Info. Proc.
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4. **Solution**: An Arithmetic with Trees
5. **"Academic Field"**: Machine Implementable Mathematics $\subset$ Constructive Mathematics
6. **Contribution to Community**: A C++ class library that builds on GNU Scientific Library, C-XSC 2.0, templatized

Extending Arithmetic:

reals → intervals → mapped partitions of interval

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4. – **by** exploiting the *algebraic structure of partitions formed by finite-rooted-binary (frb) trees*
5. – **thereby** provide algorithms for several *inclusion algebras over frb tree partitions*

arithmetic from intervals to their frb-tree partitions

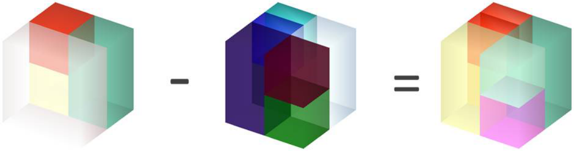

Figure: Arithmetic with coloured spaces.

arithmetic from intervals to their frb-tree partitions

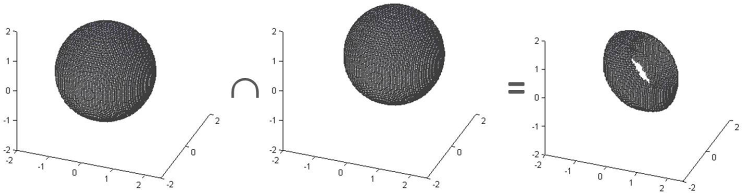

Figure: Intersection of enclosures of two hollow spheres.

arithmetic from intervals to their frb-tree partitions

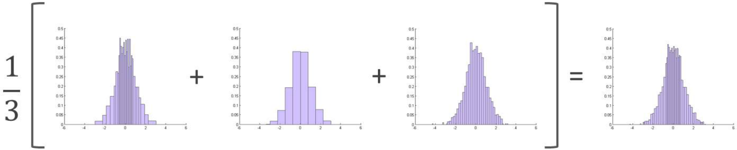

Figure: Histogram averaging.

Why Mapped Regular pavings (MRPs)?

MRPs allow any arithmetic defined over elements in $\mathbb{Y}$ to be extended point-wise to $\mathbb{Y}$-MRPs.

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MRPs allow any arithmetic defined over elements in $\mathbb{Y}$ to be extended point-wise to $\mathbb{Y}$-MRPs.

1. Arithmetic on piece-wise constant functions and interval-valued functions;
2. Exploiting the tree-based structure to obtain interval enclosures of real-valued functions efficiently
3. Statistical set-processing operations like marginal density, conditional density and highest coverage regions, visualization, etc

An RP tree a root interval $\boldsymbol{x}_\rho \in \mathbb{IR}^d$

The regularly paved boxes of $\boldsymbol{x}_\rho$ can be represented by nodes of finite rooted binary (frb-trees) of geometric group theory

An operation of bisection on a box is equivalent to performing the operation on its corresponding node in the tree:

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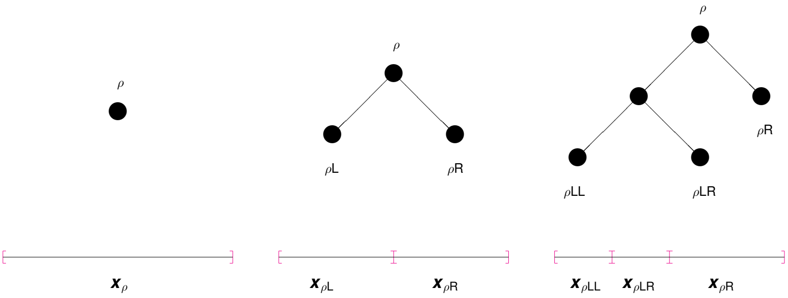
Leaf boxes of RP tree partition the root interval $\boldsymbol{x}_\rho \in \mathbb{IR}^1$



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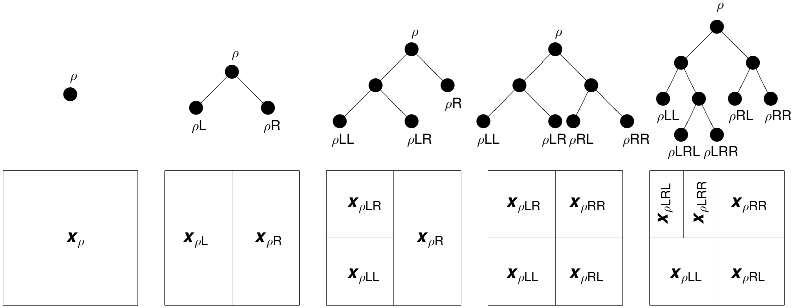
Leaf boxes of RP tree partition the root interval $\boldsymbol{x}_\rho \in \mathbb{IR}^2$



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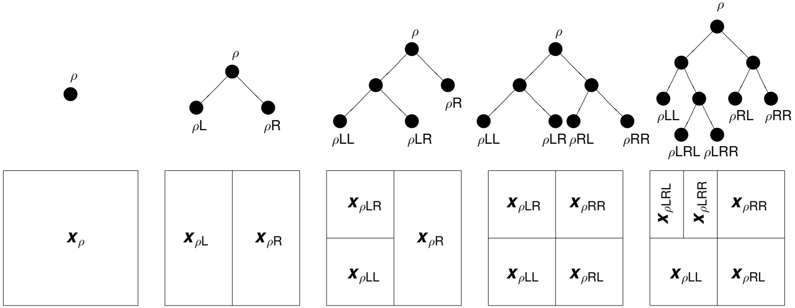
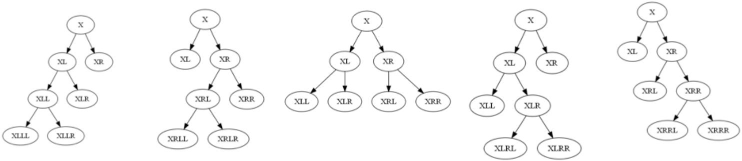


By this "RP Peano's curve" frb-trees encode paritions of $\boldsymbol{x}_\rho \in \mathbb{IR}^d$

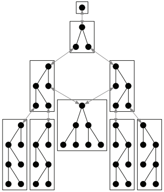
Algebraic Structure and Combinatorics of RPs

Leaf-depth encoded RPs

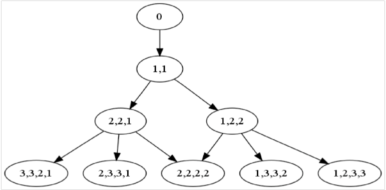


(3, 3, 2, 1) (1, 3, 3, 2) (2, 2, 2, 2) (2, 3, 3, 1) (1, 2, 3, 3)

There are C_k RPs with k splits

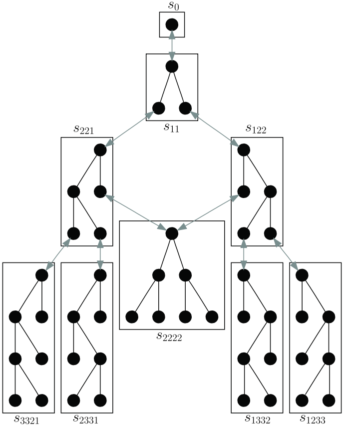


$$
\begin{aligned}
C_0 &= 1\\
C_1 &= 1\\
C_2 &= 2\\
C_3 &= 5\\
C_4 &= 14\\
C_5 &= 42\\
\ldots &= \ldots\\
C_k &= \frac{(2k)!}{(k+1)!k!}\\
\ldots &= \ldots\\
C_{15} &= 9694845\\
\ldots &= \ldots\\
C_{20} &= 6564120420\\
\ldots &= \ldots
\end{aligned}
$$



Hasse (transition) Diagram of Regular Pavings

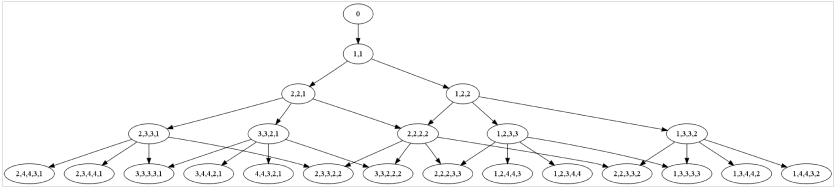
Transition diagram over $\mathbb{S}_{0:3}$ with split/reunion operations



RS, W.Taylor and G.Teng, Catalan Coefficients, Sequence A185155 in The On-Line Encyclopedia of Integer Sequences, 2012, `http://oeis.org`

Hasse (transition) Diagram of Regular Pavings

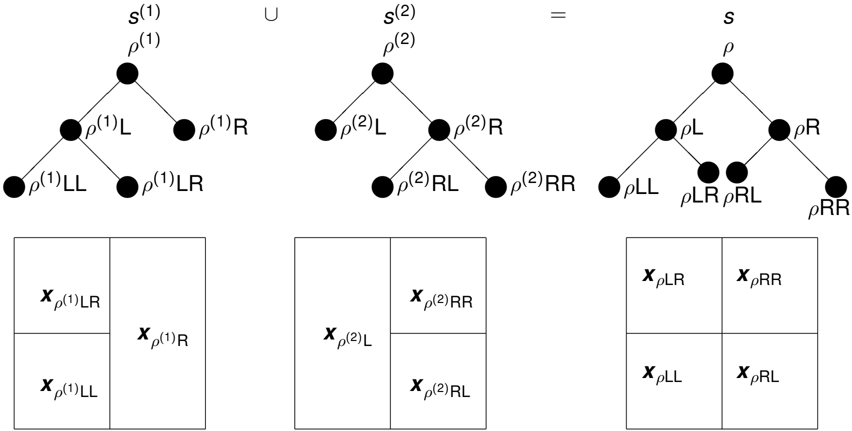
Transition diagram over $\mathbb{S}_{0:4}$ with split/reunion operations



1. The above state space is denoted by $\mathbb{S}_{0:4}$
2. Number of RPs with k splits is the Catalan number C_k
3. There is more than one way to reach a RP by k splits
4. Randomized enclosure algorithms are Markov chains on $\mathbb{S}_{0:\infty}$

RPs are closed under union operations

$s^{(1)} \cup s^{(2)} = s$ is union of two RPs $s^{(1)}$ and $s^{(2)}$ of $\boldsymbol{x}_\rho \in \mathbb{R}^2$.



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Lemma 1: The algebraic structure of frb-trees (underlying Thompson's group) is closed under union operations.

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Proof: by a "transparency overlay process" argument (cf. Meier 2008).

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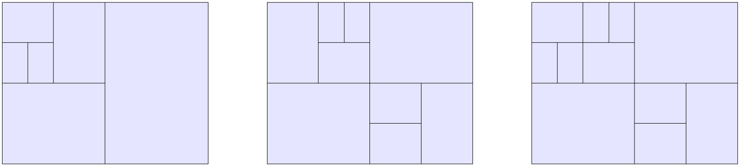

Algorithm 1: `RPUnion`$(\rho^{(1)}, \rho^{(2)})$

input : Root nodes $\rho^{(1)}$ and $\rho^{(2)}$ of RPs $s^{(1)}$ and $s^{(2)}$, respectively, with root box $\boldsymbol{x}_{\rho^{(1)}} = \boldsymbol{x}_{\rho^{(2)}}$

output : Root node ρ of RP $s = s^{(1)} \cup s^{(2)}$

if `IsLeaf`$(\rho^{(1)})$ & `IsLeaf`$(\rho^{(2)})$ **then**
    $\rho \leftarrow$ `Copy`$(\rho^{(1)})$
    return ρ
end

else if !`IsLeaf`$(\rho^{(1)})$ & `IsLeaf`$(\rho^{(2)})$ **then**
    $\rho \leftarrow$ `Copy`$(\rho^{(1)})$
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end

else if `IsLeaf`$(\rho^{(1)})$ & !`IsLeaf`$(\rho^{(2)})$ **then**
    $\rho \leftarrow$ `Copy`$(\rho^{(2)})$
    return ρ
end

else
    !`IsLeaf`$(\rho^{(1)})$ & !`IsLeaf`$(\rho^{(2)})$
end
Make ρ as a node with $\boldsymbol{x}_\rho \leftarrow \boldsymbol{x}_{\rho^{(1)}}$
Graft onto ρ as left child the node `RPUnion`$(\rho^{(1)}\mathrm{L}, \rho^{(2)}\mathrm{L})$
Graft onto ρ as right child the node `RPUnion`$(\rho^{(1)}\mathrm{R}, \rho^{(2)}\mathrm{R})$
return ρ

Note: this is not the minimal union of the (Boolean mapped) RPs of Jaulin et. al. 2001

Dfn: Mapped Regular Paving (MRP)

- Let $s \in \mathbb{S}_{0:\infty}$ be an RP with root node ρ and root box $\boldsymbol{x}_\rho \in \mathbb{IR}^d$

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- and let $\mathbb{Y}$ be a non-empty set.

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- Let $\mathbb{V}(s)$ and $\mathbb{L}(s)$ denote the sets all nodes and leaf nodes of s, respectively.
- Let $f : \mathbb{V}(s) \to \mathbb{Y}$ map each node of s to an element in $\mathbb{Y}$ as follows:

$$\{\rho\mathsf{v} \mapsto f_{\rho\mathsf{v}} : \rho\mathsf{v} \in \mathbb{V}(s), f_{\rho\mathsf{v}} \in \mathbb{Y}\} \ .$$

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- Such a map f is called a $\mathbb{Y}$*-mapped regular paving (*$\mathbb{Y}$*-MRP)*.

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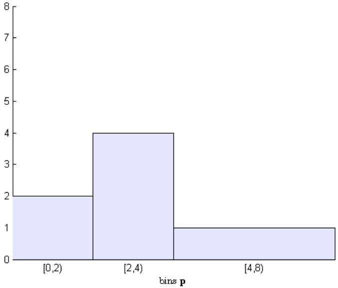
$$\{\rho\mathsf{v} \mapsto f_{\rho\mathsf{v}} : \rho\mathsf{v} \in \mathbb{V}(s), f_{\rho\mathsf{v}} \in \mathbb{Y}\} \ .$$
- Such a map f is called a *$\mathbb{Y}$-mapped regular paving ($\mathbb{Y}$-MRP)*.
- Thus, a $\mathbb{Y}$-MRP f is obtained by augmenting each node $\rho\mathsf{v}$ of the RP tree s with an additional data member $f_{\rho\mathsf{v}}$.

Examples of $\mathbb{Y}$-MRPs

If $\mathbb{Y} = \mathbb{R}$

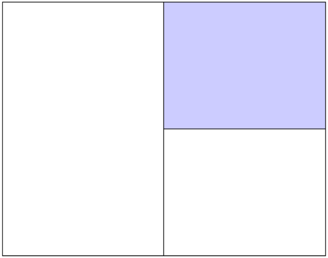
$\mathbb{R}$-MRP over s_{221} with $x_\rho = [0, 8]$



Examples of $\mathbb{Y}$-MRPs

If $\mathbb{Y} = \mathbb{B}$

$\mathbb{B}$-MRP over s_{122} with $x_\rho = [0, 1]^2$ (e.g. Jaulin et. al. 2001)

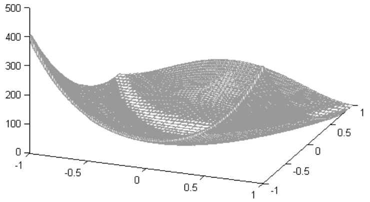


Examples of $\mathbb{Y}$-MRPs

If $\mathbb{Y} = \mathbb{IR}$

– frb tree representation for interval inclusion algebra

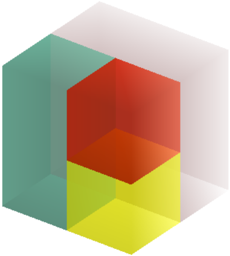
$\mathbb{IR}$-MRP enclosure of the Rosenbrock function with $x_\rho = [-1, 1]^2$



Examples of $\mathbb{Y}$-MRPs

If $\mathbb{Y} = [0,1]^3$
– R G B colour maps

$[0,1]^3$-MRP over s_{3321} with $x_\rho = [0,1]^3$

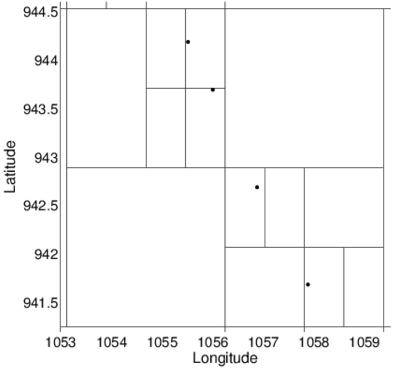


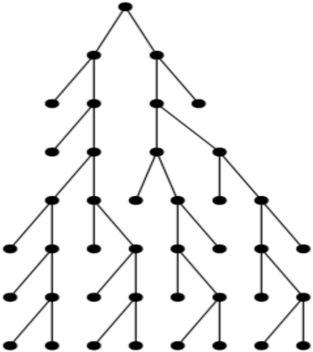
Examples of $\mathbb{Y}$-MRPs

If $\mathbb{Y} = \mathbb{Z}_+ := \{0, 1, 2, ...\}$

– radar-measured aircraft trajectory data

$\mathbb{Z}_+$-MRP trajectory of an aircraft and its tree



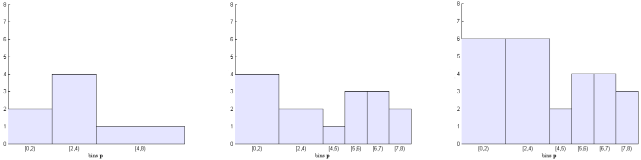


$\mathbb{Y}$-MRP Arithmetic

If $\star : \mathbb{Y} \times \mathbb{Y} \to \mathbb{Y}$ then we can extend $\star$ point-wise to two $\mathbb{Y}$-MRPs *f* and *g* with root nodes $\rho^{(1)}$ and $\rho^{(2)}$ via `MRPOperate`$(\rho^{(1)}, \rho^{(2)}, \star)$.

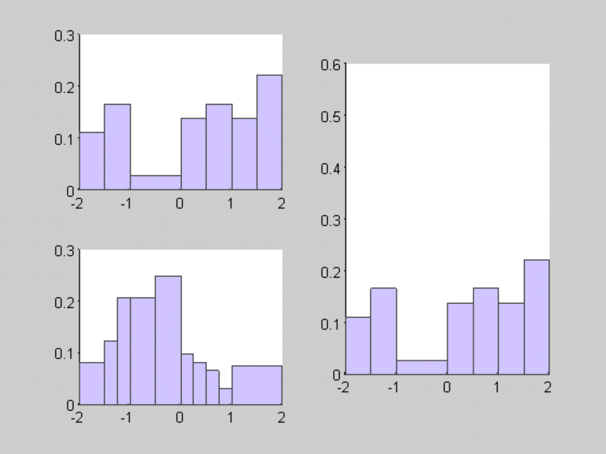
This is done using `MRPOperate`$(\rho^{(1)}, \rho^{(2)}, +)$

f *g* $f + g$



$\mathbb{R}$-MRP Addition by MRPOperate$(\rho^{(1)}, \rho^{(2)}, +)$

adding two piece-wise constant functions or $\mathbb{R}$-MRPs



Algorithm 2: MRPOperate$(\rho^{(1)}, \rho^{(2)}, \star)$

input : two root nodes $\rho^{(1)}$ and $\rho^{(2)}$ with same root box $\boldsymbol{x}_{\rho^{(1)}} = \boldsymbol{x}_{\rho^{(2)}}$ and binary operation $\star$.

output : the root node ρ of $\mathbb{Y}$-MRP $h = f \star g$.

Make a new node ρ with box and image
$\boldsymbol{x}_\rho \leftarrow \boldsymbol{x}_{\rho^{(1)}}$; $h_\rho \leftarrow f_{\rho^{(1)}} \star g_{\rho^{(2)}}$

if IsLeaf$(\rho^{(1)})$ & !IsLeaf$(\rho^{(2)})$ **then**
- Make temporary nodes L′, R′
- $\boldsymbol{x}_{\mathrm{L}'} \leftarrow \boldsymbol{x}_{\rho^{(1)}\mathrm{L}}$; $\boldsymbol{x}_{\mathrm{R}'} \leftarrow \boldsymbol{x}_{\rho^{(1)}\mathrm{R}}$
- $f_{\mathrm{L}'} \leftarrow f_{\rho^{(1)}}$, $f_{\mathrm{R}'} \leftarrow f_{\rho^{(1)}}$
- Graft onto ρ as left child the node MRPOperate(L′, $\rho^{(2)}$L, $\star$)
- Graft onto ρ as right child the node MRPOperate(R′, $\rho^{(2)}$R, $\star$)

end

else if !IsLeaf$(\rho^{(1)})$ & IsLeaf$(\rho^{(2)})$ **then**
- Make temporary nodes L′, R′
- $\boldsymbol{x}_{\mathrm{L}'} \leftarrow \boldsymbol{x}_{\rho^{(2)}\mathrm{L}}$; $\boldsymbol{x}_{\mathrm{R}'} \leftarrow \boldsymbol{x}_{\rho^{(2)}\mathrm{R}}$
- $g_{\mathrm{L}'} \leftarrow g_{\rho^{(2)}}$, $g_{\mathrm{R}'} \leftarrow g_{\rho^{(2)}}$
- Graft onto ρ as left child the node MRPOperate($\rho^{(1)}$L, L′, $\star$)
- Graft onto ρ as right child the node MRPOperate($\rho^{(1)}$R, R′, $\star$)

end

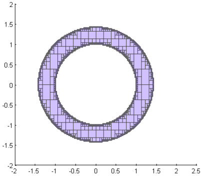
else if !IsLeaf$(\rho^{(1)})$ & !IsLeaf$(\rho^{(2)})$ **then**
- Graft onto ρ as left child the node MRPOperate($\rho^{(1)}$L, $\rho^{(2)}$L, $\star$)
- Graft onto ρ as right child the node MRPOperate($\rho^{(1)}$R, $\rho^{(2)}$R, $\star$)

end
return ρ

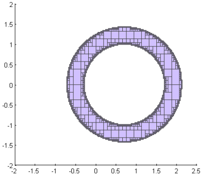
$\mathbb{B}$-MRP arithmetic

Two Boolean-mapped regular pavings A_1 and A_2 and Boolean arithmetic operations with $+$ for set union, $-$ for symmetric set difference, $\times$ for set intersection, and $\div$ for set difference.

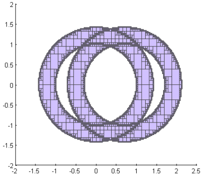
A_1



A_2



$A1 + A2$



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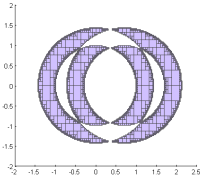
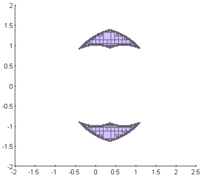
Two Boolean-mapped regular pavings A_1 and A_2 and Boolean arithmetic operations with $+$ for set union, $-$ for symmetric set difference, $\times$ for set intersection, and $\div$ for set difference.

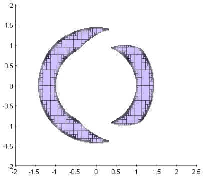
$A_1 - A2$



$A_1 \times A_2$



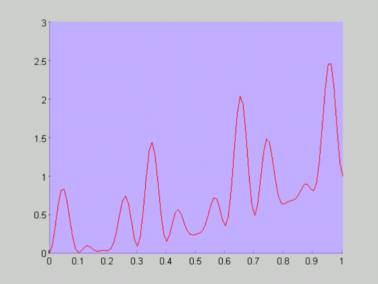
$A_1 \div A_2$



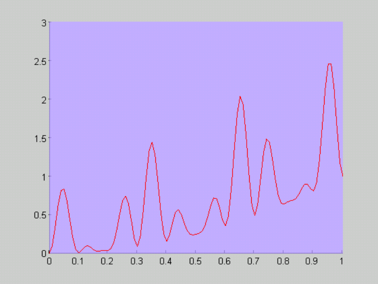
Example – Prioritised Splitting

inclusion function: $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{x}^2 + (\boldsymbol{x} + 1)\sin(10\pi\boldsymbol{x})^2\cos(3\pi\boldsymbol{x})^2$
priority function: $\psi(\rho\mathsf{v}) = \mathrm{vol}\,(\rho\mathsf{v})\mathrm{wid}\,(\boldsymbol{g}(\boldsymbol{x}_{\rho\mathsf{v}}))$

To 50 leaves by
$\texttt{RPQEnclose}^{\bigtriangledown}(\rho, \boldsymbol{g}, \psi, \bar{\ell} = 50)$



To 100 leaves by
$\texttt{RPQEnclose}^{\bigtriangledown}(\rho, \boldsymbol{g}, \psi, \bar{\ell} = 100)$



Algorithm 3: $\texttt{RPQEnclose}^{\triangledown}(\rho, \boldsymbol{g}, \psi, \bar{\ell})$

input : ρ, the root node of $\mathbb{IR}$-MRP $\boldsymbol{f}$ with RP s, root box $\boldsymbol{x}_\rho$ and $\boldsymbol{f}_\rho = \boldsymbol{g}(\boldsymbol{x}_\rho)$,
$\psi : \mathbb{L}(s) \rightarrow \mathbb{R}$ such that
$\psi(\rho\mathsf{v}) = \mathrm{vol}\,(\boldsymbol{x}_{\rho\mathsf{v}})\,(\boldsymbol{g}(\boldsymbol{x}_{\rho\mathsf{v}}) - 0.5\,(\boldsymbol{g}(\boldsymbol{x}_{\rho\mathsf{vL}}) + \boldsymbol{g}(\boldsymbol{x}_{\rho\mathsf{vR}})))$,
$\bar{\ell}$ the maximum number of leaves.

output : $\boldsymbol{f}$ with modified RP s such that $|\mathbb{L}(s)| = \bar{\ell}$

if $|\mathbb{L}(s)| < \bar{\ell}$ **then**

$$\rho\mathsf{v} \leftarrow \texttt{random\_sample}\left(\underset{\rho\mathsf{v} \in \mathbb{L}(s)}{\operatorname{argmax}} \psi(\rho\mathsf{v})\right)$$

Split $\rho\mathsf{v}$: $\triangledown(\rho\mathsf{v}) = \{\rho\mathsf{vL}, \rho\mathsf{vR}\}$ `// split the sampled node`

$\boldsymbol{f}_{\rho\mathsf{vL}} \leftarrow \boldsymbol{g}(\square(\boldsymbol{x}_{\rho\mathsf{vL}}))$

$\boldsymbol{f}_{\rho\mathsf{vR}} \leftarrow \boldsymbol{g}(\square(\boldsymbol{x}_{\rho\mathsf{vL}}))$

$\texttt{RPQEnclose}^{\triangledown}(\rho, \psi, \bar{\ell})$

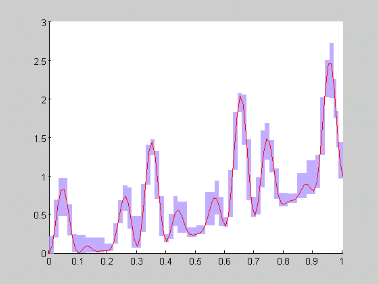
end

Example - Prioritised Splitting Continued

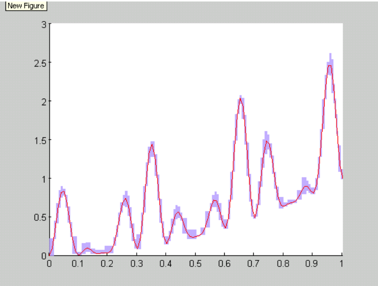
inclusion function: $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{x}^2 + (\boldsymbol{x}+1)\sin(10\pi\boldsymbol{x})^2\cos(3\pi\boldsymbol{x})^2$

priority function: $\psi(\rho\mathsf{v}) = \mathrm{vol}\,(\rho\mathsf{v})\mathrm{wid}\,(\boldsymbol{g}(\boldsymbol{x}_{\rho\mathsf{v}}))$

To 50 leaves by $\texttt{RPQEnclose}^{\nabla}(\rho, \boldsymbol{g}, \psi, \overline{\ell} = 50)$



To 100 leaves by $\texttt{RPQEnclose}^{\nabla}(\rho, \boldsymbol{g}, \psi, \overline{\ell} = 100)$



Can we get tighter enclosures using only 50 leaves by propagating the interval hull of 100-leaved $\mathbb{IR}$-MRP up the tree and then doing a prioritised merging of the cherries?

Hull Propagate up the tree via $\texttt{HullPropagate}(\rho)$

Algorithm 4: $\texttt{HullPropagate}(\rho)$

input : ρ, the root node of $\mathbb{IR}$-MRP $\boldsymbol{f}$ with RP s.
output : Modify input MRP $\boldsymbol{f}$.

if $!\texttt{IsLeaf}(\rho)$ **then**
    $\texttt{HullPropagate}(\rho\text{L})$
    $\texttt{HullPropagate}(\rho\text{R})$
    $\boldsymbol{f}_{\rho} \leftarrow \boldsymbol{f}_{\rho\text{L}} \sqcup \boldsymbol{f}_{\rho\text{R}}$
end

By calling $\texttt{HullPropagate}(\rho)$ on our $\mathbb{IR}$-MRP of Example constructed by $\texttt{RPQEnclose}^{\triangledown}(\rho, \boldsymbol{g}, \psi, \overline{\ell} = 100)$ we would have tightened the range enclosures of $\boldsymbol{g}$ in the internal nodes.

Prioritised Merging via $\texttt{RPQEnclose}^{\triangle}(\rho, \psi, \bar{\ell}')$

Algorithm 5: $\texttt{RPQEnclose}^{\triangle}(\rho, \psi, \bar{\ell}')$

input : ρ, the root node of $\mathbb{IR}$-MRP $\boldsymbol{f}$ with RP s, box $\boldsymbol{x}_\rho$,
$\psi : \mathbb{C}(s) \to \mathbb{R}$ as $\psi(\rho\mathsf{v}) = \mathrm{vol}\,(\boldsymbol{x}_{\rho\mathsf{v}})\,(\boldsymbol{f}_{\rho\mathsf{v}} - 0.5\,(\boldsymbol{f}_{\rho\mathsf{vL}} + \boldsymbol{f}_{\rho\mathsf{vR}}))$,
$\bar{\ell}'$ the maximum number of leaves.

output : modified $\boldsymbol{f}$ with RP s such that $|\mathbb{L}(s)| = \bar{\ell}'$ or $\mathbb{C}(s) = \emptyset$.

if $|\mathbb{L}(s)| \geq \bar{\ell}'$ & $\mathbb{C}(s) \neq \emptyset$ **then**

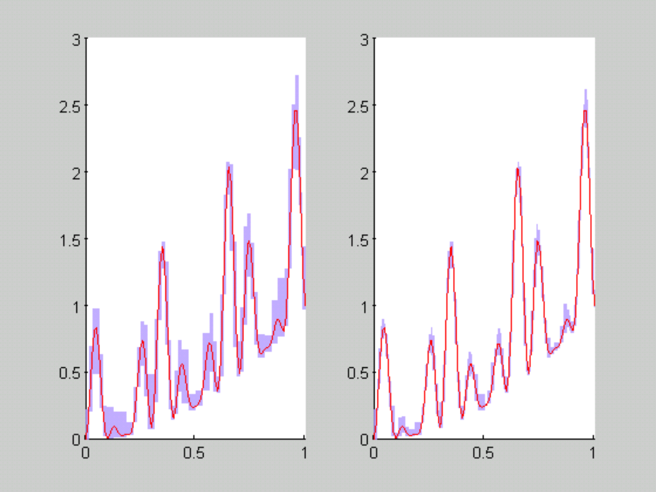
```
    ρv ← random_sample(argmin_{ρv∈C(s)} ψ(ρv))        // choose a
    random node with smallest ψ
    Prune(ρL)
    Prune(ρR)
    RPQEnclose△(ρ, ψ, ℓ̄')
```

end

Example – Split, Propogating & Prune

Yes we can!

$\texttt{RPQEnclose}^{\triangledown}(\rho, \boldsymbol{g}, \psi, \bar{\ell} = 100); \texttt{HullPropagate}(\rho); \texttt{RPQEnclose}^{\triangle}(\rho, \psi, \bar{\ell}' = 50)$

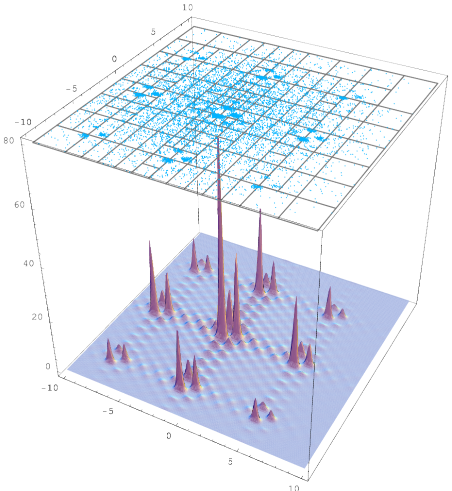


Statistical Applications

- “Nonparametric Density Estimation” with massive metric data streams
- Stat. Operations: Coverage, Marginal integral and Slice
- Memory-efficient Arithmetic for Air Traffic Co-trajectories
- Life Science Appl.: Animal Migration Track
- Bold untried Idea: Set-valued Arithmetic for Geospatial Data (Global EQ data)

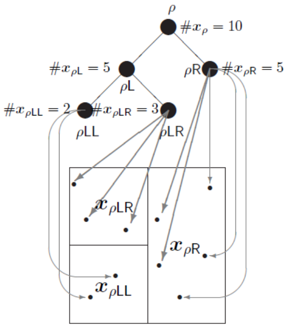
Nonparametric Density Estimation

Problem: Take samples from an unknown density f and consistently reconstruct f

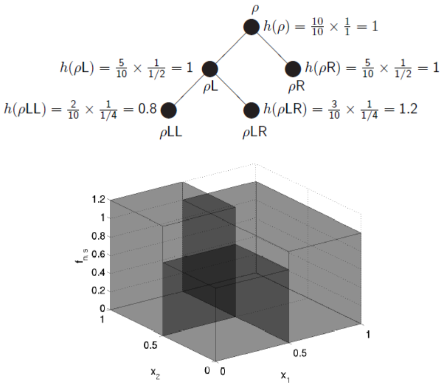


Nonparametric Density Estimation

Approach: Use statistical regular paving to get $\mathbb{R}$-MRP data-adaptive histogram



(a) An SRP tree and its constituents.

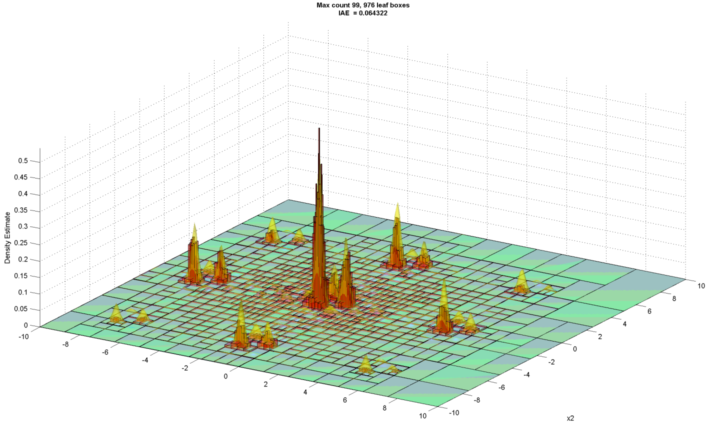


(b) An SRP histogram and its tree.

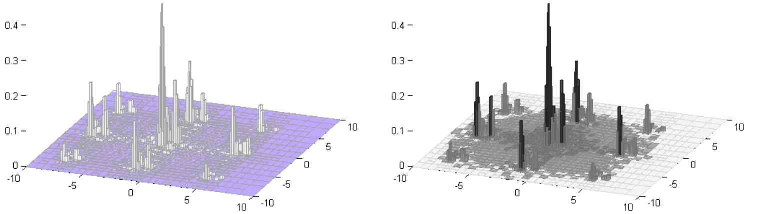
Nonparametric Density Estimation

Solution: $\mathbb{R}$-MRP histogram averaging allows us to produce a consistent Bayesian estimate of the density (up to 10 dimensions)

(Teng, Harlow, Lee and S., *ACM Trans. Mod. & Comp. Sim.*, [r. 2] 2012)

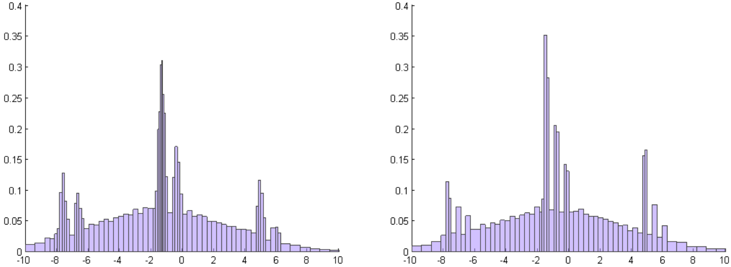


Coverage, Marginal & Slice Operators of $\mathbb{R}$-MRP



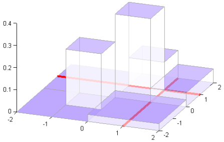
$\mathbb{R}$-MRP approximation to Levy density and its coverage regions with $\alpha = 0.9$ (light gray), $\alpha = 0.5$ (dark gray) and $\alpha = 0.1$ (black)

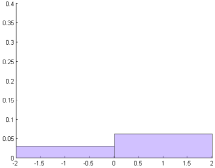
Coverage, Marginal & Slice Operators of $\mathbb{R}$-MRP

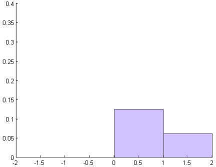


Marginal densities $f^{\{1\}}(x_1)$ and $f^{\{2\}}(x_2)$ along each coordinate of $\mathbb{R}$-MRP approximation

Coverage, Marginal & Slice Operators of $\mathbb{R}$-MRP





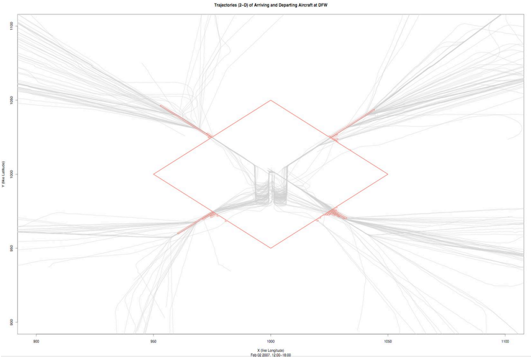


The slices of a simple $\mathbb{R}$-MRP in 2D

Air Traffic "Arithmetic" $\rightarrow$ dynamic air-space configuration

(G. Teng, K. Kuhn and RS, *J. Aerospace Comput., Inf. & Com.*, 9:1, 14–25, 2012.)

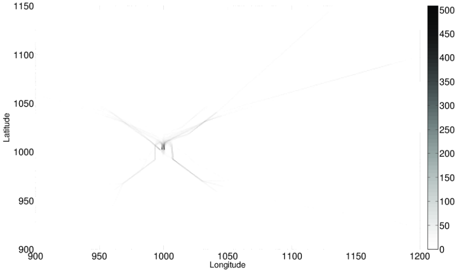
On a Good Day



Air Traffic “Arithmetic” $\rightarrow$ dynamic air-space configuration

(G. Teng, K. Kuhn and RS, *J. Aerospace Comput., Inf. & Com.*, 9:1, 14–25, 2012.)

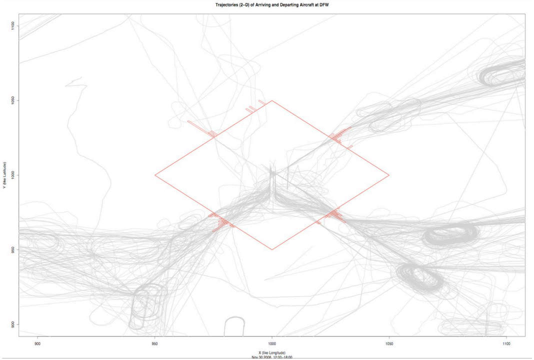
$\mathbb{Z}_+$-MRP On a Good Day



Air Traffic “Arithmetic” $\rightarrow$ dynamic air-space configuration

(G. Teng, K. Kuhn and RS, *J. Aerospace Comput., Inf. & Com.*, 9:1, 14–25, 2012.)

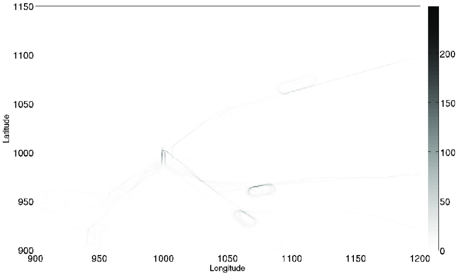
On a Bad Day



Air Traffic “Arithmetic” $\rightarrow$ dynamic air-space configuration

(G. Teng, K. Kuhn and RS, *J. Aerospace Comput., Inf. & Com.*, 9:1, 14–25, 2012.)

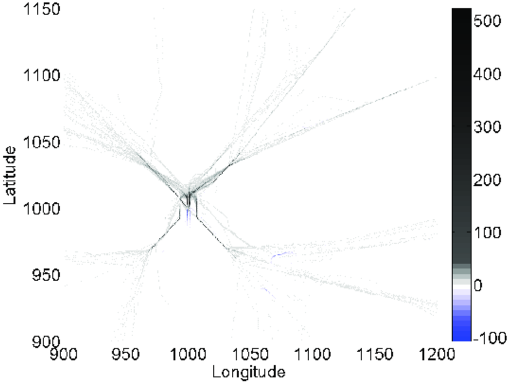
$\mathbb{Z}_+$-MRP On a Bad Day



Air Traffic "Arithmetic" → dynamic air-space configuration

(G. Teng, K. Kuhn and RS, *J. Aerospace Comput., Inf. & Com.*, 9:1, 14–25, 2012.)

$\mathbb{Z}_{+}$-MRP pattern for Good Day − Bad Day



Conclusions

- $\mathbb{Y}$-MRPs provide frb-tree partition arithmetic
- $\mathbb{IY}$-MRPs allow efficient arithmetic for Neumaier's inclusion algebras
- $\mathbb{IY}$ can be $\mathbb{IR}$ for $\boldsymbol{f} : \mathbb{IR}^d \to \mathbb{IR}$
- $\mathbb{IY}$ can be $\mathbb{IR}^m$ for $\boldsymbol{f} : \mathbb{IR}^d \to \mathbb{IR}^m$
- $\mathbb{IY}$ can be $(\mathbb{IR}, \mathbb{IR}^m, \mathbb{IR}^{m^2})$ for range, gradient & Hessian of $\boldsymbol{f} : \mathbb{IR}^d \to \mathbb{IR}$
- Other obvious extensions include arithmetic over Taylor polynomial inclusion algebras
- In general the domain and range of $\boldsymbol{f}$ can be complete lattices with intervals and bisection operations
- We have seen several statistical applications of $\mathbb{Y}$-MRPs
- CODE: *mrs: a C++ class library for statistical set processing* by Bycroft, Harlow, Sainudiin, Teng and York.

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Thank you!